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The Approximate Loebl-Komlós-Sós Conjecture

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Abstract

We prove the following version of the Loebl-Komlós-Sós Conjecture: For every $\alpha > 0$ there exists a number k_0 such that for every $k > k_0$ every n-vertex graph G with at least $(\frac{1}{2} + \alpha)n$ vertices of degree at least $(1 + \alpha)k$ contains each tree T of order k as a subgraph.

The method to prove our result follows a strategy common to approaches which employ the Szemerédi Regularity Lemma: we decompose the graph G, find a suitable combinatorial structure inside the decomposition, and then embed the tree T into G using this structure. However, the decomposition given by the Regularity Lemma is not of help when G is sparse. To surmount this shortcoming we use a more general decomposition technique: each graph can be decomposed into vertices of huge degree, regular pairs (in the sense of the Regularity Lemma), and two other objects each exhibiting certain expansion properties.

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1 Introduction

1.1 Statement of the problem

We provide an approximate solution of the Loebl-Komlós-Sós Conjecture. This is a problem in extremal graph theory which fits the classical form *Does a certain density condition imposed on a graph guarantee a certain subgraph?* Classical results of this type include Dirac's Theorem which determines the minimum degree threshold for containment of a Hamilton cycle, or Mantel's Theorem which determines the average degree threshold for containment of a triangle. Indeed, most of these extremal problems are formulated in terms of the minimum or average degree of the host graph.

We investigate density conditions which guarantee that a host graph contains *each* tree of order k. The greedy tree-embedding strategy shows that minimum degree more of than $k-2$ is a sufficient condition. Further, this bound is best possible as any $(k-2)$-regular graph avoids the k-vertex star. However, Erdős and Sós conjectured that the minimum degree condition can be relaxed to an average degree one still giving the same conclusion.

Conjecture 1.1 (Erdős-Sós Conjecture 1963)**.** *Let G be a graph of average degree greater than $k-2$. Then G contains each tree of order k as a subgraph.*

A solution of the Erdős-Sós Conjecture for all k bigger than an absolute constant was announced by Ajtai, Komlós, Simonovits, and Szemerédi in the early 1990's. In a similar spirit, Loebl, Komlós, and Sós conjectured that a *median degree* of $k-1$ or more is sufficient for containment of any tree of order k. By median degree we mean the degree of a vertex in the middle of the ordered degree sequence.

Conjecture 1.2 (Loebl-Komlós-Sós Conjecture 1995 [EFLS95])**.** *Suppose that G is an n-vertex graph with at least $n/2$ vertices of degree more than $k-2$. Then G contains each tree of order k.*

We discuss in detail Conjectures 1.1 and 1.2 in Section 1.3. Here, we just state the main result of the paper, an approximate solution of the Loebl-Komlós-Sós Conjecture.

Theorem 1.3 (Main result)**.** *For every $\alpha > 0$ there exists k_0 such that for any $k > k_0$ we have the following. Each n-vertex graph G with at least $(\frac{1}{2}+\alpha)n$ vertices of degree at least $(1+\alpha)k$ contains each T tree of order k.*

1.2 Regularity lemma and dense graph theory

The Szemerédi Regularity Lemma has been a major tool in extremal graph theory for three decades. It provides an approximate representation of a graph with a so-called *cluster graph*. This cluster graph representation is the key for graph-containment problems. The usual strategy here is that instead of solving the original problem one focuses on a modified simpler problem in the cluster graph.

The applicability of the Szemerédi Regularity Lemma is, however, limited to *dense graphs*, i.e., graphs that contain a substantial proportion of all possible edges. Luckily enough many graphs arising in extremal graph theory are dense, as for example those coming from Dirac's and Mantel's Theorem above. But, while the proofs of these two results are elementary many of their extensions rely on the Regularity Lemma.

While the theory of dense graphs is well understood due to the Szemerédi Regularity Lemma, no such tool is available for sparse graphs. A regularity type representation of general (possibly sparse) graphs is one of the most important goals of contemporary discrete mathematics. By such a representation we mean an approximation of the input graph by a structure of bounded complexity carrying enough of the important information about the graph.

A central tool in the proof of Theorem 1.3 is a structural decomposition of the graph $G_{\triangleright T1.3}$. This decomposition — which we call *sparse decomposition* — applies to any graph whose average degree is bigger than an absolute constant. The sparse decomposition provides a partition of any graph into vertices of huge degrees and into a bounded degree part. The bounded degree part is further decomposed into dense regular pairs, an edge set with certain expander-like properties, and a vertex set which is expanding in a different way (we shall give a more precise description in Section 1.5). This kind of decomposition was first used by Ajtai, Komlós, Simonovits, and Szemerédi in their yet unpublished work on the Erdős-Sós Conjecture.

In the case of dense graphs the sparse decomposition produces a Szemerédi regularity partition, and thus the decomposition lemma (Lemma 4.13) extends the Szemerédi Regularity Lemma. But, the interesting setting for the Decomposition Lemma are sparse graphs. Being sparse, these graphs may be expected to contain less interesting substructures than dense graphs, and so, it comes as no surprise that the output of Lemma 4.13 in this setting is less useful than a Szemerédi regularity partition for dense graphs. If we think of graph containment problems, the applicability of Lemma 4.13 seems to be limited to simple structures as trees.

1.3 Loebl-Komlós-Sós Conjecture and Erdős-Sós Conjecture

Let us first introduce some notation. We say that H *embeds* in a graph G and write $H \subseteq G$ if H is a (not necessarily induced) subgraph of G. The associated map $\phi : V(H) \to V(G)$ is called an *embedding of H in G*. More generally, for a graph class $\mathcal{H}$ we write $\mathcal{H} \subseteq G$ if $H \subseteq G$ for every $H \in \mathcal{H}$. Let $\mathbf{trees}(k)$ be the class of all trees of order k.

Conjecture 1.2 is dominated by two parameters: one quantifies the number of vertices of 'large' degree, and the other tells us how large this degree should actually be. Strengthening either of these bounds sufficiently, the conjecture becomes trivial. [1]

On the other hand, one may ask whether lower bounds would suffice. For the bound $k-2$, this is not the case, since stars of order k require a vertex of degree at least $k-1$ in the host graph. As

[1] Indeed, if we replace $n/2$ with n, then any tree of order k can be embedded greedily. Also, if we replace $k-2$ with $4k-4$, then G, being a graph of average degree at least $2k-2$, has a subgraph G' of minimum degree at least $k-1$. Again we can greedily embed any tree of order k.

$\frac{k}{2} - 1$

$\frac{k}{2} + 1$

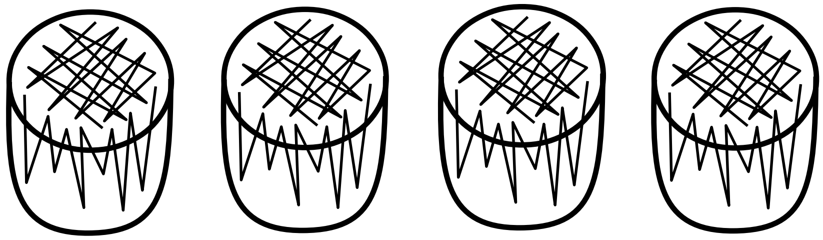

Figure 1.1: An extremal graph for the Loebl-Komlós-Sós Conjecture.

for the bound $n/2$, the following example shows that this number cannot be decreased much.

First, assume that n is even, and that $n = k$. Let G^* be obtained from the complete graph on n vertices by deleting all edges inside a set of $\frac{n}{2} + 1$ vertices. It is easy to check that G^* does not contain the path[2] $P_k \in \mathbf{trees}(k)$. Now, taking the union of several disjoint copies of G^* we obtain examples for other values of n. (And adding a small complete component we can get to *any* value of n.) See Figure 1.1 for an illustration.

However, we do not know of any example attaining the exact bound $n/2$. Thus it might be possible to lower the bound $n/2$ from Conjecture 1.2 to the one attained in our example above:

Conjecture 1.4. *Let $k \in \mathbb{N}$ and let G be a graph on n vertices, with more than $\frac{n}{2} - \lfloor \frac{n}{k} \rfloor - (n \bmod k)$ vertices of degree at least $k-1$. Then $\mathbf{trees}(k) \subseteq G$.*

It might even be that if n/k is far from integrality, a slightly lower bound on the number of vertices of large degree still works (see [Hla, HP]).

Several partial results concerning Conjecture 1.2 have been obtained; let us briefly summarize the major ones. Two main directions can be distinguished among those results that prove the conjecture for special classes of graphs: either one places restrictions on the host graph, or on the class of trees to be embedded. Of the latter type is the result by Bazgan, Li, and Woźniak [BLW00], who proved the conjecture for paths. Also, Piguet and Stein [PS08] proved that Conjecture 1.2 is true for trees of diameter at most 5, which improved earlier results of Barr and Johansson [BJ] and Sun [Sun07].

Restrictions on the host graph have led to the following results. Soffer [Sof00] showed that Conjecture 1.2 is true if the host graph has girth at least 7. Dobson [Dob02] proved the conjecture for host graphs whose complement does not contain a $K_{2,3}$. This has been extended by Matsumoto and Sakamoto [MS] who replace the $K_{2,3}$ with a slightly larger graph.

A different approach is to solve the conjecture for special values of k. One such case, known as the Loebl conjecture, or also as the $(n/2$–$n/2$–$n/2)$–Conjecture, is the case $k = n/2$. Ajtai, Komlós, and Szemerédi [AKS95] solved an approximate version of this conjecture, and later Zhao [Zha11] used a refinement of this approach to prove the sharp version of the conjecture for large graphs.

An approximate version of Conjecture 1.2 for dense graphs, that is, for k linear in n, was proved by Piguet and Stein [PS12]. Let us take this opportunity to introduce a useful notation. Write

[2] In general, G^* does not contain any tree $T \in \mathbf{trees}(k)$ which has an equitable two-coloring.

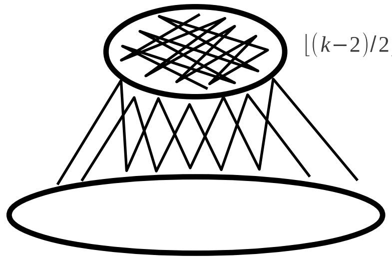

Figure 1.2: An almost extremal graph for the Erdős-Sós Conjecture.

$\mathbf{LKS}(n,k,\alpha)$ for the class of all n-vertex graphs with at least $(\frac{1}{2}+\alpha)n$ vertices of degrees at least $(1+\alpha)k$. With this notation Conjecture 1.2 states that every graph in $\mathbf{LKS}(n,k,0)$ contains every tree from $\mathbf{trees}(k+1)$.

Theorem 1.5 (Piguet-Stein [PS12])**.** *For any $q>0$ and $\alpha>0$ there exists a number n_0 such that for any $n>n_0$ and $k>qn$ the following holds. If $G\in\mathbf{LKS}(n,k,\alpha)$ then $\mathbf{trees}(k+1)\subseteq G$.*

This result was proved using the regularity method. Adding stability arguments, Hladký and Piguet [HP], and independently Cooley [Coo09] proved Conjecture 1.2 for large dense graphs.

Theorem 1.6 (Hladký-Piguet [HP], Cooley [Coo09])**.** *For any $q>0$ there exists a number $n_0=n_0(q)$ such that for any $n>n_0$ and $k>qn$ the following holds. If $G\in\mathbf{LKS}(n,k,0)$ then $\mathbf{trees}(k+1)\subseteq G$.*

Let us now turn our attention to the Erdős-Sós Conjecture. It is particularly important to compare the structure of the respective extremal graph with the extremal graphs for the Loebl-Kómlos-Sós Conjecture. The Erdős-Sós Conjecture 1.1 is best possible whenever $n(k-2)$ is even. Indeed, in that case it suffices to consider a $(k-2)$-regular graph. This is a graph with average degree exactly $k-2$ which does not contain the star of order k. Even when the star (which in a sense is a pathological tree) is excluded from the considerations, we can — at least when $k-1$ divides n — consider a disjoint union of $\frac{n}{k-1}$ cliques K_{k-1}. This graph contains *no* tree from $\mathbf{trees}(k)$.

There is another important graph with many edges which does not contain for example the path P_k, depicted in Figure 1.2. This graph has $\frac{1}{2}(k-2)n-O(k^2)$ edges when k is even and $\frac{1}{2}(k-3)n-O(k^2)$ edges otherwise, and therefore gets close to the conjectured bound when $k\ll n$. Apart from the already mentioned announced breakthrough by Ajtai, Komlós, Simonovits, and Szemerédi, work on this conjecture includes [BD96, Hax01, MS, SW97, Woź96].

Both Conjectures 1.2 and Conjecture 1.1 have an important application in Ramsey theory. Each of them implies that the Ramsey number of two trees $T_{k+1}\in\mathbf{trees}(k+1)$, $T_{\ell+1}\in\mathbf{trees}(\ell+1)$ is bounded by $R(T_{k+1},T_{\ell+1})\leqslant k+\ell+1$. Actually more is implied: Any 2-edge-colouring of $K_{k+\ell+1}$ contains either *all* trees in $\mathbf{trees}(k+1)$ in red, or *all* trees in $\mathbf{trees}(\ell+1)$ in blue.

The bound $R(T_{k+1},T_{\ell+1})\leqslant k+\ell+1$ is almost tight only for certain types of trees: Harary [Har72] showed $R(S_k,S_\ell)=k+\ell-2-\varepsilon$ for stars $S_k\in\mathbf{trees}(k)$, $S_\ell\in\mathbf{trees}(\ell)$, where $\varepsilon\in\{0,1\}$

depends on the parity of k and ℓ. On the other hand, Gerencsér and Gyárfás [GG67] showed $R(P_k, P_\ell) = \max\{k, \ell\} + \left\lfloor \frac{\min\{k,\ell\}}{2} \right\rfloor - 1$ for paths $P_k \in \mathbf{trees}(k)$, $P_\ell \in \mathbf{trees}(\ell)$. Haxell, Łuczak, and Tingley confirmed asymptotically [HLT02] that the discrepancy of the Ramsey bounds for trees depends on their balancedness, at least when the maximum degrees of the trees considered are moderately bounded.

1.4 Related tree containment problems

Trees in random graphs. To complete the picture of research involving tree containment problems we mention two rich and vivid (and also closely connected) areas: trees in random graphs, and trees in expanding graphs. The former area is centered around the following question: *What is the probability threshold $p = p(n)$ for the Erdős-Rényi random graph $G_{n,p}$ to contain asymptotically almost surely (a.a.s.) each tree/all trees from a given class $\mathcal{F}_n$ of trees?* Note that there is a difference between containing "each tree" and "all trees" as the error probabilities for missing individual trees might sum up.

Most research focused on containment of spanning trees, or almost spanning trees. The only well-understood case is when $\mathcal{F}_n = \{P_{k_n}\}$ is a path. The threshold $p = \frac{(1+o(1))\ln n}{n}$ for appearance of a spanning path (i.e., $k_n = n$) was determined by Komlós and Szemerédi [KS83], and independently by Bollobás [Bol84]. Note that this threshold is the same as the threshold for a weaker property for connectedness. We should also mention a previous result of Pósa [Pós76] which determined the order of magnitude of the threshold, $p = \Theta(\frac{\ln n}{n})$. The heart of Pósa's proof, the celebrated rotation-extension technique, is an argument about expanding graphs, and indeed many other results about trees in random graphs exploit the expansion properties of $G_{n,p}$ in the first place.

The threshold for the appearance of almost spanning paths in $G_{n,p}$ was determined by Fernandez de la Vega [FdlV79] and independently by Ajtai, Komlós, and Szemerédi [AKS81]. Their results say that a path of length $(1-\varepsilon)n$ appears a.a.s. in $G_{n,\frac{C}{n}}$ for $C = C(\varepsilon)$ sufficiently large. This behavior extends to bounded degree trees. Indeed, Alon, Krivelevich, and Sudakov [AKS07] proved that $G_{n,\frac{C}{n}}$ (for a suitable $C = C(\varepsilon, \Delta)$) a.a.s. contains all trees of order $(1-\varepsilon)n$ with maximum degree at most Δ (the constant C was later improved in [BCPS10]).

Let us now turn to spanning trees in random graphs. It is known [AKS07] that a.a.s. $G_{n,\frac{C\ln n}{n}}$ contains a single spanning tree T with bounded maximum degree and linearly many leaves. This result can be reduced to the main result of [AKS07] regarding almost spanning trees quite easily. The constant C can be taken $C = 1 + o(1)$, as was shown recently by Hefetz, Krivelevich, and Szabó [HKS]; obviously this is best possible. The same result also applies to trees that contain a path of linear length whose vertices all have degree two. A breakthrough in the area was achieved by Krivelevich [Kri10] who gave an upper bound on the threshold $p = p(n, \Delta)$ for embedding a single spanning tree of a given maximum degree Δ. This bound is essentially tight for $\Delta = n^c$, $c \in (0, 1)$. Even though the argument in [Kri10] is not difficult, it relies on a deep result of Johansson, Kahn and Vu [JKV08] about factors in random graphs.

Trees in expanders. By an expander graph we mean a graph with a large Cheeger constant, i.e., a graph which satisfies a certain isoperimetric property. As indicated above, random graphs are very good expanders, and this is the main motivation for studying tree containment problems in expanders. Another motivation comes from studying the universality phenomenon. Here the goal is to construct sparse graphs which contain all trees from a given class, and expanders are natural candidates for this. The study of sparse tree-universal graphs is a remarkable area by itself which brings challenges both in probabilistic and explicit constructions. For example, Bhatt, Chung, Leighton, and Rosenberg [BCLR89] give an explicit construction of a graph with only $O_\Delta(n)$ edges which contains all n-vertex trees with maximum degree at most Δ. More recently, Johannsen, Krivelevich, and Samotij [JKS12] showed a number of universality results for spanning trees of maximum degree $\Delta = \Delta(n)$ both for random graphs, and for expanders. For example, they show universality for this class of each graph with a large Cheeger constant that satisfies a certain connectivity condition.

Friedman and Pippenger [FP87] extended Pósa's rotation-extension technique from paths to trees by and found many applications (e.g. [HK95, Hax01, BCPS10]). Sudakov and Vondrák [SV10] use tree-indexed random walks to embed trees in $K_{s,t}$-free graphs (this property implies expansion); a similar approach is employed by Benjamini and Schramm [BS97] in the setting of infinite graphs.

In our proof of Theorem 1.3, embedding trees in expanders play a crucial role, too. However, our notion of expansion is very different from those studied previously. (Actually, we introduce two, very different, notions in Definitions 4.2 and 4.6.)

Minimum degree conditions for spanning trees. Recall that the tight min-degree condition for containment of a general spanning tree T in an n-vertex graph G is the trivial one, $\mathrm{deg}^{\min}(G) \geqslant n-1$. However, the only tree which requires this bound is the star. This indicates that this threshold can be lowered substantially if we have a control of $\mathrm{deg}^{\max}(T)$. Szemerédi and his collaborators [KSS01, CLNGS10] showed that this is indeed the case, and obtained tight min-degree bounds for certain ranges of $\mathrm{deg}^{\max}(T)$. For example, if $\mathrm{deg}^{\max}(T) \leqslant n^{o(1)}$, then $\mathrm{deg}^{\min}(G) \geqslant (\frac{1}{2} + o(1))n$ is a sufficient condition. (Note that G may become disconnected close to this bound.)

1.5 Overview of the proof of Theorem 1.3

The structure of the proof of Theorem 1.3 resembles the proof of the dense case, Theorem 1.5. We obtain an approximate representation — called *sparse decomposition* — of the graph $G_{\triangleright\mathrm{T}1.3}$. Then we find a suitable combinatorial structure inside the sparse decomposition. Finally, we embed the tree $T_{\triangleright\mathrm{T}1.3}$ into $G_{\triangleright\mathrm{T}1.3}$ using this structure.

First, let us give a short outline of how the manuscript is structured. We use Sections 2–8 to introduce all tools necessary for the proof of Theorem 1.3, which is given in a relatively short form in Section 9. Section 10 contains some concluding remarks.

The preparation for the proof of the main theorem during Sections 2–8 starts with introducing some general preliminaries in Section 2. Then, in Section 3, the tree $T_{\triangleright\mathrm{T}1.3}$ is pre-processed by

being cut into tiny subtrees, with few connecting vertices.

Sections 4–7 deal with the graph $G_{\triangleright\mathrm{T}1.3}$. First, Section 4 introduces the notion of the sparse decomposition which captures an approximate representation of $G_{\triangleright\mathrm{T}1.3}$. (Such a sparse decomposition exists for all graphs, and in a sense is comparable with the Szemerédi regularity partition.) Then, in Sections 5 and 6 we gather more structural information, specifically using the properties of graphs from $\mathbf{LKS}(n,k,\alpha)$. This finally leads to several possible "configurations", as we call them, presented in Section 7. These configurations give a quite precise description of $G_{\triangleright\mathrm{T}1.3}$ that can be used for tree embedding.

Finally, in Section 8 we introduce techniques for embedding small trees in a graph, based on the configurations we found in Section 7. In addition to the standard filling-up-a-regular-pair technique usually employed in conjunction with the regularity method, we employ several other techniques adapted to the diverse other parts of our sparse decomposition.

A scheme of the proof is given in Figure 1.3.

Let us describe now the key ingredients of the proof in more detail. The input graph $G_{\triangleright\mathrm{T}1.3} \in \mathbf{LKS}(n,k,\alpha)$ has $\Theta(kn)$ edges.[3] Recall that the Szemerédi Regularity Lemma gives an approximation of dense graphs in which $o(n^2)$ edges are neglected. In analogy, the sparse decomposition captures all but at most $o(kn)$ edges. The vertices of $G_{\triangleright\mathrm{T}1.3}$ are partitioned into vertices of degrees $\gg k$ and vertices of degree $O(k)$. Further, the graph induced by the latter set is split into regular pairs (in the sense of the Szemerédi Regularity Lemma) with clusters of sizes $\Theta(k)$, and into two additional parts which exhibit certain expansion properties (the expansion properties of these two parts are different). The vertices of huge degrees, the regular pairs, and the two expanding parts form the sparse decomposition of $G_{\triangleright\mathrm{T}1.3}$. It is well-known that regular pairs are suitable for embedding small trees. In Section 8 we work out techniques for embedding small trees in each of the three remaining parts of the sparse decomposition.

Tree-embedding results in the dense setting (e.g. Theorem 1.5) rely on finding a matching structure in the cluster graph. Indeed, this allows one to distribute different parts of the tree in the matching edges. In analogy, in Lemma 6.1 we find a structure which combines all four components of the sparse decomposition, and which we call the *rough structure*. Not only all parts of the sparse decomposition are contained in the rough structure, but also, on top of these, an additional object, which we call a *s*emiregular matching. This is found with the help of Lemma 5.10, a step which we call "augmenting a matching". The necessity of this step is discussed in detail in Section 6.2.

However, the rough structure is not immediately suitable for embedding $T_{\triangleright\mathrm{T}1.3}$, and we shall further refine it in Section 7.7 to one of ten *configurations*, denoted by $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$. Obtaining these configurations from the rough structure is based on pigeonhole-type arguments such as: if there are many edges between two sets, and few "kinds" of edges, then many of the edges are of the same kind. The different kinds of edges come from the sparse decomposition (and allow for different kinds

[3]Indeed, an easy counting argument gives that $e(G_{\triangleright\mathrm{T}1.3}) \geqslant kn/4$. On the other hand, we can assume that $e(G_{\triangleright\mathrm{T}1.3}) < kn$, as otherwise $G_{\triangleright\mathrm{T}1.3}$ contains a subgraph with minimum degree at least k, and the assertion of Theorem 1.3 follows.

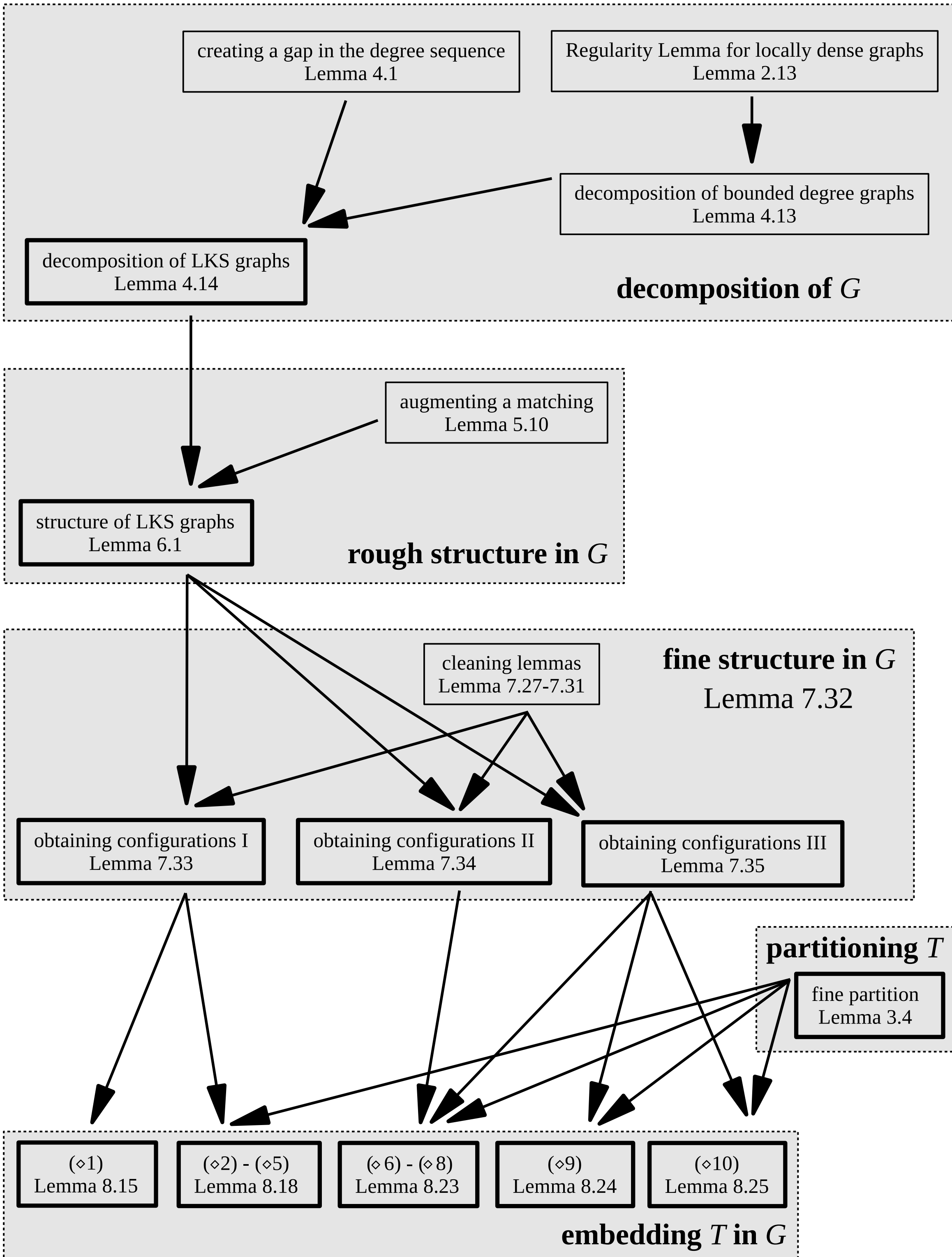

Figure 1.3: Structure of the proof of Theorem 1.3.

of embedding techniques). Just "homogenizing" the situation by restricting to one particular kind is not enough, we also need to employ certain "cleaning lemmas" — Lemmas 7.27–7.31. A simplest such lemma would be that a graph with many edges contains a subgraph with a large minimum degree; the latter property evidently being more directly applicable for a sequential embedding of a tree. The actual cleaning lemmas we use are complex extensions of this simple idea.

Finally, in Section 8, we show how to embed the tree $T_{\triangleright\mathrm{T}1.3}$. This is done by first establishing some elementary embedding lemmas for small subtrees in Section 8.3, and then combine these in Section 8.4 for each of the cases ($\diamond\mathbf{1}$)–($\diamond\mathbf{10}$) to yield an embedding of the entire tree $T_{\triangleright\mathrm{T}1.3}$.

2 Notation and preliminaries

In this section we recall some standard terminology and introduce some further specific notation. We also state some basic results from graph theory.

2.1 Notation

The set $\{1, 2, \dots, n\}$ of the first n positive integers is denoted by $[n]$. Suppose that we have a nonempty set A, and $\mathcal{X}$ and $\mathcal{Y}$ each partition A. Then $\boxplus$ denotes the coarsest common refinement of $\mathcal{X}$ and $\mathcal{Y}$, i.e.,

$$\mathcal{X} \boxplus \mathcal{Y} := \{X \cap Y \ : \ X \in \mathcal{X}, Y \in \mathcal{Y}\} \setminus \{\emptyset\} \ .$$

We frequently employ indexing by many indices. We write superscript indices in parentheses (such as $a^{(3)}$), as opposed to notation of powers (such as a^3). We use sometimes subscript to refer to parameters appearing in a fact/lemma/theorem. For example $\alpha_{\triangleright\mathrm{T}1.3}$ refers to the parameter α from Theorem 1.3. We omit rounding symbols when this does not affect the correctness of the arguments.

We use lower case greek letters to denote small positive constants. The exception is the letter ϕ which is reserved for embedding of a tree T in a graph G, $\phi : V(T) \to V(G)$. The capital greek letters are used for large constants.

2.2 Basic graph theory notation

All graphs considered in this paper are finite, undirected, without multiple edges, and without self-loops. We write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G, respectively. Further, $v(G) = |V(G)|$ is the order of G, and $e(G) = |E(G)|$ is its number of edges. If $X, Y \subseteq V(G)$ are two, not necessarily disjoint, sets of vertices we write $e(X)$ for the number of edges induced by X, and $e(X, Y)$ for the number of ordered pairs $(x, y) \in X \times Y$ such that $xy \in E(G)$. In particular, note that $2e(X) = e(X, X)$.

For a graph G, a vertex $v \in V(G)$ and a set $U \subseteq V(G)$, we write $\deg(v)$ and $\deg(v, U)$ for the degree of v, and for the number of neighbours of v in U, respectively. We write $\deg^{\min}(G)$ for the minimum degree of G, $\deg^{\min}(U) := \min\{\deg(u) \ : \ u \in U\}$, and $\deg^{\min}(V_1, V_2) = \min\{\deg(u, V_2) \ :$

$u \in V_1\}$ for two sets $V_1, V_2 \subseteq V(G)$. Similar notation is used for the maximum degree, denoted by $\deg^{\max}(G)$. The neighbourhood of a vertex v is denoted by $\mathrm{N}(v)$. We set $\mathrm{N}(U) := \bigcup_{u \in U} \mathrm{N}(u)$. The symbol $-$ is used for two graph operations: if $U \subseteq V(G)$ is a vertex set then $G - U$ is the subgraph of G induced by the set $V(G) \setminus U$. If $H \subseteq G$ is a subgraph of G then the graph $G - H$ is defined on the vertex set $V(G)$ and corresponds to deletion of edges of H from G.

A subgraph $H \subseteq G$ of a graph G is called *spanning* if $V(H) = V(G)$.

The *null graph* is the unique graph on zero vertices, while any graph with zero edges is called *empty*.

A family $\mathcal{A}$ of pairwise disjoint subsets of $V(G)$ is an *ℓ-ensemble in G* if $|A| \geqslant \ell$ for each $A \in \mathcal{A}$. We say that $\mathcal{A}$ is *inside* X (or *outside* Y) if $A \subseteq X$ (or $A \cap Y = \emptyset$) for each $A \in \mathcal{A}$.

If T is a tree and $r \in V(T)$, then the pair (T, r) is a *rooted tree* with root r. We then write $V_{\mathrm{odd}}(T, r) \subseteq V(T)$ for the set of vertices of T of odd distance from r. Analogously we define $V_{\mathrm{even}}(T, r)$. Note that $r \in V_{\mathrm{even}}(T, r) \subseteq V(T)$. The distance between two vertices v_1 and v_2 in a tree is denoted by $\mathrm{dist}(v_1, v_2)$.

We next give two simple facts about the number of leaves in a tree. These have already appeared in [Zha11] and in [HP] (and most likely in some more classic texts as well). Nevertheless, for completeness we shall include their proofs here.

Fact 2.1. *Let T be a tree with color-classes X and Y, and $v(T) \geqslant 2$. Then the set X contains at least $|X| - |Y| + 1$ leaves of T.*

Proof. Root T at an arbitrary vertex $r \in Y$. Let I be the set of internal vertices of T that belong to X. Each $v \in I$ has at least one immediate successor in the tree order induced by r. These successors are distinct for distinct $v \in I$ and all lie in $Y \setminus \{r\}$. Thus $|I| \leqslant |Y| - 1$. The claim follows. □

Fact 2.2. *Let T be a tree with ℓ vertices of degree at least three. Then T has at least $\ell + 2$ leaves.*

Proof. Let D_1 be the set of leaves, D_2 the set of vertices of degree two and D_3 be the set of vertices of degree of at least three. Then

$$2(|D_1| + |D_2| + |D_3|) - 2 = 2v(T) - 2 = 2e(T) = \sum_{v \in V(T)} \deg(v) \geqslant |D_1| + 2|D_2| + 3|D_3| \ ,$$

and the statement follows. □

For the next lemma, note that for us, the minimum degree of the null graph is ∞.

Lemma 2.3. *For all $\ell, n \in \mathbb{N}$, every n-vertex graph G contains a (possibly empty) subgraph G' such that $\deg^{\min}(G') \geqslant \ell$ and $e(G') \geqslant e(G) - (\ell - 1)n$.*

Proof. We construct the graph G' by sequentially removing vertices of degree less than ℓ from the graph G. In each step we remove at most $\ell - 1$ edges. Thus the statement follows. □

We finish this section with stating the Gallai-Edmonds matching theorem. A graph H is called *factor-critical* if $H - v$ has a perfect matching for each $v \in V(H)$. The following statement is a fundamental result in matching theory. See [LP86], for example.

Theorem 2.4 (Gallai-Edmonds matching theorem)**.** *Let H be a graph. Then there exist a set $Q \subseteq V(H)$ and a matching M of size $|Q|$ in H such that*

1) every component of $H - Q$ is factor-critical, and

2) M matches every vertex in Q to a different component of $H - Q$.

The set Q in Theorem 2.4 is often referred to as a *separator*.

2.3 LKS-minimal graphs

Given a graph G, denote by $\mathbb{S}_{\eta,k}(G)$ the set of those vertices of G that have degree less than $(1+\eta)k$ and by $\mathbb{L}_{\eta,k}(G)$ the set of those vertices of G that have degree at least $(1+\eta)k$.[4] Thus the sizes of the sets $\mathbb{S}_{\eta,k}(G)$ and $\mathbb{L}_{\eta,k}(G)$ are what specifies the membership to $\mathbf{LKS}(n, k, \eta)$ (which we had defined as the class of all n-vertex graphs with at least $(\frac{1}{2}+\eta)n$ vertices of degrees at least $(1+\eta)k$).

Define $\mathbf{LKSmin}(n, k, \eta)$ as the set of all graphs $G \in \mathbf{LKS}(n, k, \eta)$ that are edge-minimal with respect to the membership in $\mathbf{LKS}(n, k, \eta)$. In order to prove Theorem 1.3 it suffices to restrict our attention to graphs from $\mathbf{LKSmin}(n, k, \eta)$, and this is why we introduce the class. Let us collect some properties of graphs in $\mathbf{LKSmin}(n, k, \eta)$ which follow directly from the definition.

Fact 2.5. *For any graph $G \in \mathbf{LKSmin}(n, k, \eta)$ the following is true.*

1. *$\mathbb{S}_{\eta,k}(G)$ is an independent set.*
2. *All the neighbours of every vertex $v \in V(G)$ with $\deg(v) > \lceil (1+\eta)k \rceil$ have degree exactly $\lceil (1+\eta)k \rceil$.*
3. *$|\mathbb{L}_{\eta,k}(G)| \leqslant \lceil (1/2+\eta)n \rceil + 1$.*

Observe that every edge in a graph $G \in \mathbf{LKSmin}(n, k, \eta)$ is incident to at least one vertex of degree exactly $\lceil (1+\eta)k \rceil$. This gives the following inequality.

$$e(G) \leqslant \lceil (1+\eta)k \rceil \, |\mathbb{L}_{\eta,k}(G)| \overset{\text{F2.5(3.)}}{\leqslant} \lceil (1+\eta)k \rceil \left(\left\lceil \left(\frac{1}{2} + \eta \right) n \right\rceil + 1 \right) < kn \, . \tag{2.1}$$

(The last inequality is valid under the additional mild assumption that, say, $\eta < \frac{1}{20}$ and $n > k > 20$. This can be assumed throughout the paper.)

Definition 2.6. *Let $\mathbf{LKSsmall}(n, k, \eta)$ be the class of those graphs $G \in \mathbf{LKS}(n, k, \eta)$ for which we have the following three properties:*

[4] "$\mathbb{S}$" stands for "small", and "$\mathbb{L}$" for "large".

1. *All the neighbours of every vertex $v \in V(G)$ with $\deg(v) > \lceil (1+2\eta)k \rceil$ have degrees at most $\lceil (1+2\eta)k \rceil$.*

2. *All the neighbours of every vertex of $\mathbb{S}_{\eta,k}(G)$ have degree exactly $\lceil (1+\eta)k \rceil$.*

3. *We have $e(G) \leqslant kn$.*

Observe that the graphs from $\mathbf{LKSsmall}(n, k, \eta)$ also satisfy 1., and a quantitatively somewhat weaker version of 2. of Fact 2.5. This suggests that in some sense $\mathbf{LKSsmall}(n, k, \eta)$ is a good approximation of $\mathbf{LKSmin}(n, k, \eta)$.

As said, we will prove Theorem 1.3 only for graphs from $\mathbf{LKSmin}(n, k, \eta)$. However, it turns out that the structure of $\mathbf{LKSmin}(n, k, \eta)$ is too rigid. In particular, $\mathbf{LKSmin}(n, k, \eta)$ is not closed under discarding a small amount of edges during our cleaning procedures. This is why the class $\mathbf{LKSsmall}(n, k, \eta)$ comes into play: starting with a graph in $\mathbf{LKSmin}(n, k, \eta)$ we perform some initial cleaning and obtain a graph that lies in $\mathbf{LKSsmall}(n, k, \eta/2)$. We then heavily use its structural properties from Definition 2.6 throughout the proof.

2.4 Regular pairs

In this section we introduce the notion of regular pairs which is central for Szemerédi's Regularity Lemma and its extension which we discuss in Section 2.5. We also list some simple properties of regular pairs.

Given a graph H and a pair (U, W) of disjoint sets $U, W \subseteq V(H)$ the *density of the pair* (U, W) is defined as

$$\mathrm{d}(U, W) := \frac{e(U, W)}{|U||W|} .$$

Similarly, for a bipartite graph G with colour classes U, W we talk about its *bipartite density* $\mathrm{d}(G) = \frac{e(G)}{|U||W|}$. For a given $\varepsilon > 0$, a pair (U, W) of disjoint sets $U, W \subseteq V(H)$ is called an *ε-regular pair* if $|\mathrm{d}(U, W) - \mathrm{d}(U', W')| < \varepsilon$ for every $U' \subseteq U$, $W' \subseteq W$ with $|U'| \geqslant \varepsilon|U|$, $|W'| \geqslant \varepsilon|W|$. If the pair (U, W) is not ε-regular, then we call it *ε-irregular*. A stronger notion than regularity is that of super-regularity which we recall now. A pair (A, B) is *(ε, γ)-super-regular* if it is ε-regular, and we have $\deg^{\min}(A, B) \geqslant \gamma|B|$, and $\deg^{\min}(B, A) \geqslant \gamma|A|$. Note that then (A, B) has bipartite density at least γ.

We list two useful and well-known properties of regular pairs.

Fact 2.7. *Suppose that (U, W) is an ε-regular pair of density d. Let $U' \subseteq W, W' \subseteq W$ be sets of vertices with $|U'| \geqslant \alpha|U|$, $|W'| \geqslant \alpha|W|$, where $\alpha > \varepsilon$. Then the pair (U', W') is a $2\varepsilon/\alpha$-regular pair of density at least $d - \varepsilon$.*

Fact 2.8. *Suppose that (U, W) is an ε-regular pair of density d. Then all but at most $\varepsilon|U|$ vertices $v \in U$ satisfy $\deg(v, W) \geqslant (d - \varepsilon)|W|$.*

The following fact states a simple relation between the density of a (not necessarily regular) pair and the densities of its subpairs.

Fact 2.9. *Let $H = (U, W; E)$ be a bipartite graph of $\mathrm{d}(U, W) \geqslant \alpha$. Suppose that the sets U and W are partitioned into sets $\{U_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$, respectively. Then at most $\beta e(H)/\alpha$ edges of H belong to a pair (U_i, W_j) with $\mathrm{d}(U_i, W_j) \leqslant \beta$.*

Proof. Trivially, we have

$$\sum_{i \in I, j \in J} \frac{|U_i||W_j|}{|U||W|} = 1 \ . \tag{2.2}$$

Consider a pair (U_i, W_j) of $\mathrm{d}(U_i, W_j) \leqslant \beta$. Then

$$e(U_i, W_j) \leqslant \beta |U_i||W_j| = \frac{\beta}{\alpha} \frac{|U_i||W_j|}{|U||W|} \alpha |U||W| \leqslant \frac{\beta}{\alpha} \frac{|U_i||W_j|}{|U||W|} e(U, W) \ .$$

Summing over all such pairs (U_i, W_j) and using (2.2) yields the statement. □

The next lemma asserts that if we have many ε-regular pairs (R, Q_i), then most vertices in R have approximately the total degree into the set $\bigcup_i Q_i$ that we would expect.

Lemma 2.10. *Let $Q_1, \ldots, Q_\ell$ and R be disjoint vertex sets. Suppose further that for each $i \in [\ell]$, the pair (R, Q_i) is ε-regular. Then we have*

(a) $\deg(v, \bigcup_i Q_i) \geqslant \frac{e(R, \bigcup_i Q_i)}{|R|} - \varepsilon \left|\bigcup_i Q_i\right|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$, and*

(b) $\deg(v, \bigcup_i Q_i) \leqslant \frac{e\left(R, \bigcup_i Q_i\right)}{|R|} + \varepsilon \left|\bigcup_i Q_i\right|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$.*

Proof. We prove (a), the other item is analogous. Suppose for contradiction that (a) does not hold. Without loss of generality, assume that there is a set $X \subseteq R$, $|X| > \varepsilon|R|$ such that $\frac{e(R, \bigcup Q_i)}{|R|} - \varepsilon|\bigcup Q_i| > \deg(v, \bigcup Q_i)$ for each $v \in X$. By averaging, there is an index $i \in [\ell]$ such that $\frac{|X|}{|R|} e(R, Q_i) - \varepsilon|X||Q_i| > e(X, Q_i)$, or equivalently,

$$\mathrm{d}(R, Q_i) - \varepsilon > \mathrm{d}(X, Q_i) \ .$$

This is a contradiction to the ε-regularity of the pair (R, Q_i). □

We use Lemma 2.10 to obtain the following.

Corollary 2.11. *Let $Q_1, \ldots, Q_\ell$ and R be disjoint vertex sets, each of size at most q, such that for each $i \in [\ell]$, the pair (R, Q_i) is ε-regular. Assume that more than $\varepsilon|R|$ vertices of R have degree at least x into $\bigcup Q_i$, but each $v \in R$ has neighbours in at most z of the sets Q_i. Then $\deg(v, \bigcup_i Q_i) \geqslant x - 2\varepsilon z q$ for all but at most $\varepsilon|R|$ vertices of R.*

Proof. For each $w \in R$, let $I_w \subseteq [\ell]$ be the set of those indices i for which there is at least one edge from w to Q_i. Now, by Lemma 2.10(b) there is a vertex $v \in R$ whose degree into $\bigcup_{i \in [\ell]} Q_i$ is at least x and whose degree into $\bigcup_{i \in I_v} Q_i$ is at most $\frac{e\left(R, \bigcup_{i \in I_v} Q_i\right)}{|R|} + \varepsilon \left|\bigcup_{i \in I_v} Q_i\right|$. So,

$$x \leqslant \deg(v, \bigcup_{i \in [\ell]} Q_i) = \deg(v, \bigcup_{i \in I_v} Q_i) \leqslant \frac{e\left(R, \bigcup_{i \in I_v} Q_i\right)}{|R|} + \varepsilon |\bigcup_{i \in I_v} Q_i| \leqslant \frac{e\left(R, \bigcup_{i \in I_v} Q_i\right)}{|R|} + \varepsilon z q.$$

Thus by Lemma 2.10(a) all but at most $\varepsilon|R|$ vertices of R have degree at least $x - 2\varepsilon z q$ into $\bigcup_i Q_i$. □

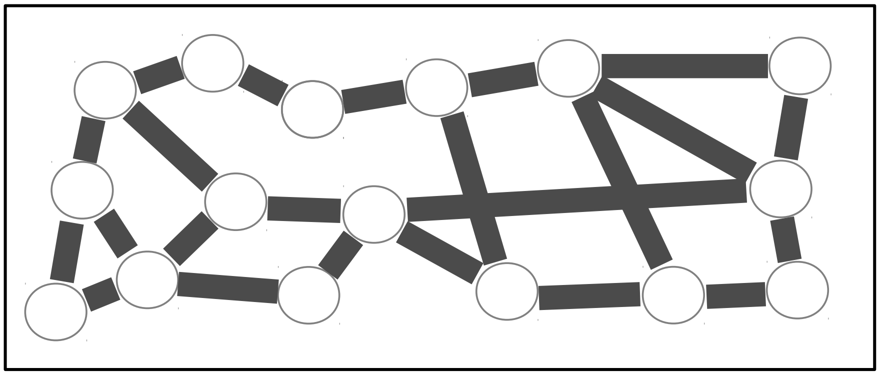

Figure 2.1: A locally dense graph as in Lemma 2.13. The sets $W_1, \ldots, W_\ell$ are depicted with grey circles. Even though there is a large number of them, each W_i is linked to only boundedly many other W_j's (at most four, in this example). Lemma 2.13 allows us to regularize all the bipartite graphs using the same system of partitions of the sets W_i.

2.5 Regularizing locally dense graphs

The Regularity Lemma [Sze78] has proved to be a powerful tool for attacking graph embedding problems; see [KO09] for a survey. We first state the lemma in its original form.

Lemma 2.12 (Regularity lemma)**.** *For all $\varepsilon > 0$ and $\ell \in \mathbb{N}$ there exist $n_0, M \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds. Let G be an n-vertex graph whose vertex set is pre-partitioned into sets $V_1, \ldots, V_{\ell'}$, $\ell' \leqslant \ell$. Then there exists a partition $U_0, U_1, \ldots, U_p$ of $V(G)$, $\ell < p < M$, with the following properties.*

1) *For every $i, j \in [p]$ we have $|U_i| = |U_j|$, and $|U_0| < \varepsilon n$.*

2) *For every $i \in [p]$ and every $j \in [\ell']$ either $U_i \cap V_j = \emptyset$ or $U_i \subseteq V_j$.*

3) *All but at most εp^2 pairs (U_i, U_j), $i, j \in [p]$, $i \neq j$, are ε-regular.*

We shall use Lemma 2.12 for auxiliary purposes only as it is helpful only in the setting of dense graphs (i.e., graphs which have n vertices and $\Omega(n^2)$ edges). This is not necessarily the case in Theorem 1.3. For this reason, we give a version of the Regularity Lemma — Lemma 2.13 below — which allows us to regularize even sparse graphs.

More precisely, suppose that we have an n-vertex graph H whose edges lie in bipartite graphs $H[W_i, W_j]$, where $\{W_1, \ldots, W_\ell\}$ is an ensemble of sets of size $\Theta(k)$. Although ℓ may be unbounded, for a fixed $i \in [\ell]$ there are only a bounded number, say m, of indices $j \in [\ell]$ such that $H[W_i, W_j]$ is non-empty. See Figure 2.1 for an example. Lemma 2.13 then allows us to regularize (in the sense of the Regularity Lemma 2.12) all the bipartite graphs $G[W_i, W_j]$ using the same partition $\{W_i^{(0)} \dot\cup W_i^{(1)} \dot\cup \ldots \dot\cup W_i^{(p_i)} = W_i\}_{i=1}^{\ell}$. Note that when $|W_i| = \Theta(k)$ for all $i \in [\ell]$ then H has at most

$$\Theta(k^2) \cdot m \cdot \ell \leqslant \Theta(k^2) \cdot m \cdot \frac{n}{\Theta(k)} = \Theta(kn)$$

edges. Thus, when $k \ll n$, this is a regularization of a sparse graph. This "sparse Regularity Lemma" is very different to that of Kohayakawa [Koh97]). Indeed, Kohayakawa's Regularity Lemma deals with graphs which have no local condensation of edges, such as subgraphs of random graphs. Consequently, the resulting regular pairs are of density $o(1)$. In contrast, Lemma 2.13 provides us with regular pairs of density $\Theta(1)$, but, on the other hand, is useful only for graphs which are locally dense.

Lemma 2.13 (Regularity Lemma for locally dense graphs)**.** *For all $m, z \in \mathbb{N}$ and $\varepsilon > 0$ there exists $q_{\mathrm{MAXCL}} \in \mathbb{N}$ such that the following is true. Suppose H and F are two graphs, $V(F) = [\ell]$ for some $\ell \in \mathbb{N}$, and $\deg^{\max}(F) \leqslant m$. Suppose that $\mathcal{Z} = \{Z_1, \ldots, Z_z\}$ is a partition of $V(H)$. Let $\{W_1, \ldots, W_\ell\}$ be a q_{MAXCL}-ensemble in H, such that for all $i, j \in [\ell]$ we have*

$$2|W_i| \geqslant |W_j| \,. \tag{2.3}$$

Then for each $i \in [\ell]$ there exists a partition $W_i^{(0)}, W_i^{(1)}, \ldots, W_i^{(p_i)}$ of the set W_i such that for all $i, j \in [\ell]$ we have

(a) $1/\varepsilon \leqslant p_i \leqslant q_{\mathrm{MAXCL}}$,

(b) $|W_i^{(i')}| = |W_j^{(j')}|$ *for each* $i' \in [p_i]$, $j' \in [p_j]$,

(c) for each $i' \in [p_i]$ there exists $x \in [z]$ such that $W_i^{(i')} \subseteq Z_x$,

(d) $\sum_i |W_i^{(0)}| < \varepsilon \sum_i |W_i|$, *and*

(e) at most $\varepsilon\,|\mathcal{Y}|$ pairs $\left(W_i^{(i')}, W_j^{(j')}\right) \in \mathcal{Y}$ form an ε-irregular pair in H, where

$$\mathcal{Y} := \left\{ \left(W_i^{(i')}, W_j^{(j')}\right) \,:\, ij \in E(F), i' \in [p_i], j' \in [p_j] \right\} \,.$$

We use Lemma 2.13 in Lemma 4.13. Lemma 4.13 is in turn the main tool in the proof of our main structural decomposition of the graph $G_{\triangleright \mathrm{T}1.3}$, Lemma 4.14. In the proof of Lemma 4.13 we decompose $G_{\triangleright \mathrm{T}1.3}$ into several parts with very different properties, and one of these parts is a locally dense graph which can be then regularized by Lemma 4.13. A similar Regularity Lemma is used in [AKSS].

The proof of Lemma 2.13 is similar to the proof of the standard Regularity Lemma 2.12, as given for example in [Sze78]. We assume the reader's familiarity with the notion of the index (a.k.a. the mean square density), and of the Index-pumping Lemma from there. We sketch the proof of Lemma 2.13 below.

Sketch of a proof of Lemma 2.13. For the sake of brevity, we omit respecting the prepartition $\mathcal{Z}$ in this sketch; this step is standard.

Before sketching a proof of the lemma, let us describe how a more naive approach fails. For each edge $ij \in E(F)$ consider a regularization of the bipartite graph $H[W_i, W_j]$, let $\{U_{i,j}^{(i')}\}_{i' \in [q_{i,j}]}$

be the partition of W_i into clusters, and let $\{U_{j,i}^{(j')}\}_{j'\in[q_{j,i}]}$ be the partition of W_j into clusters such that almost all pairs $(U_{i,j}^{(i')}, U_{j,i}^{(j')}) \subseteq (W_i, W_j)$ form an ε'-regular pair (for some ε' of our taste). We would now be done if the partition $\{U_{i,j}^{(i')}\}_{i'\in[q_{i,j}]}$ of W_i was independent of the choice of the edge ij. This however need not be the case. The natural next step would therefore be to consider the common refinement

$$\boxplus_{j:ij\in E(F)} \{U^{(i')_{i,j}}\}_{i'\in[q_{ij}]}$$

of all the obtained partitions of W_i. The pairs obtained in this way lack however any regularity properties as they are too small. Indeed, it is a notorious drawback of the Regularity Lemma that the number of clusters in the partition is enormous as a function of the regularity parameter. In our setting, this means that $q_{i,j} \gg \frac{1}{\varepsilon'}$. Thus a typical cluster $U_{i,j_1}^{(i'_1)}$ occupies on average only a $\frac{1}{q_{i,j_1}}$-fraction of the cluster $U_{i,j_2}^{(i'_2)}$, and thus already the set $U_{i,j_1}^{(i'_1)} \cap U_{i,j_2}^{(i'_2)} \subseteq U_{i,j_2}^{(i'_2)}$ is not substantial (in the sense of the regularity). The same issue arises when regularizing multicolored graphs (cf. [KS96, Theorem 1.18]). The solution is to impel the regularizations to happen in a synchronized way.

We first recall the proof of the original Regularity Lemma 2.12 which we then modify. Actually, it better suits our situation to illustrate this on a procedure which regularizes a given bipartite graph $G = (A, B; E)$. We start with arbitrary bounded partitions $\mathcal{W}_A$ and $\mathcal{W}_B$ of A and B. Sequentially, we look whether there is a witness of irregularity of $\mathcal{W}_A$ and $\mathcal{W}_B$. If there is, then the partition $\mathcal{W}_A$ and $\mathcal{W}_B$ can be refined so that the index increases. The facts that one can control the increase of the complexity of the partitions, and that the index increases substantially are the keys for guaranteeing that the iteration terminates in a bounded number of steps.

By Vizing's Theorem we can cover the edges of F by disjoint matchings $M_1, \dots, M_{m+1}$. For each $i \in [m+1]$ we shall introduce a variable ind_i. The variable ind_i is the average index of the bipartite graphs which correspond to the edges of M_i and the current partitions of the sets W_x. In each step $i \in [m+1]$, we refine simultanously partitions in all bipartite graphs $G[W_x, W_y]$ $(xy \in M_i)$ which possess witnesses of irregularity. More precisely, assume that in a certain step each set W_z is partitioned into sets $\mathcal{W}_z$. We then define

$$\begin{aligned} \mathrm{ind}_i &= \frac{1}{|M_i|} \sum_{xy\in M_i} \mathrm{ind}(\mathcal{W}_x, \mathcal{W}_y) \ , && \text{if } M_i \neq \emptyset, \text{ and} \\ \mathrm{ind}_i &= 1 \ , && \text{otherwise.} \end{aligned}$$

where ind is the usual index. The Index-pumping Lemma asserts that when refining the partition of $G[W_x, W_y]$ the value $\mathrm{ind}(\mathcal{W}_x, \mathcal{W}_y)$ increases substantially. The fact that M_i is a matching allows us to perform these simultaneous refinements without interference. It is well-known that none of ind_j $(j < i)$ did decrease during pumping ind_i up. Thus after a bounded number of steps there are no witnesses of irregularity in the graphs $G[W_x, W_y]$ $(xy \in E(H))$ with respect to the partitions $\mathcal{W}_x, \mathcal{W}_y$. This suffices to give the statement. $\square$

Usually after applying the Regularity Lemma to some graph G, one bounds the number of

edges which correspond to irregular pairs, to regular, but sparse pairs, or are incident with the exceptional sets U_0. We shall do the same for the setting of Lemma 2.13.

Lemma 2.14. *In the situation of Lemma 2.13, suppose that* $\deg^{\max}(H) \leqslant \Omega k$ *and* $e(H) \leqslant kn$*, and that each edge* $xy \in E(H)$ *is captured by some edge* $ij \in E(F)$*, i.e.,* $x \in W_i$*,* $y \in W_j$*. Moreover suppose that*

$$\mathrm{d}(W_i, W_j) \geqslant \gamma \text{ if } ij \in E(F). \tag{2.4}$$

Then all but at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma)nk$ *edges of* H *belong to regular pairs* $(W_{i'}^{(i)}, W_{j'}^{(j)})$*,* $i, j \neq 0$*, of density at least* γ^2*.*

Proof. Set $w := \min\{|W_i| : i \in V(F)\}$. By (2.4), each edge of F represents at least γw^2 edges of H. Since $e(H) \leqslant kn$ it follows that $e(F) \leqslant kn/(\gamma w^2)$. Thus, by the assumption (2.3), $\sum_{AB \in E(F)} |A||B| \leqslant e(F)(2w)^2 \leqslant \frac{4kn}{\gamma}$. Using (e) of Lemma 2.13 we get that the number of edges of H contained in ε-irregular pairs from $\mathcal{Y}$ is at most

$$\frac{4\varepsilon nk}{\gamma} . \tag{2.5}$$

Write E_1 for the set of edges of H which are incident with a vertex in $\bigcup_{i \in [\ell]} W_i^{(0)}$. Then by (d) of Lemma 2.13, and since $\deg^{\max}(H) \leqslant \Omega k$,

$$|E_1| \leqslant \varepsilon\Omega nk . \tag{2.6}$$

Let E_2 be the set of those edges of H which belong to ε-regular pairs $(W_i^{(i')}, W_j^{(j')})$ with $ij \in E(F), i' \in [p_i], j' \in [p_j]$ of density at most γ^2. We claim that

$$|E_2| \leqslant \gamma kn . \tag{2.7}$$

Indeed, because of (2.4) and by Fact 2.9 (with $\alpha_{\triangleright\mathrm{F}2.9} := \gamma$ and $\beta_{\triangleright\mathrm{F}2.9} := \gamma^2$), for each $ij \in E(F)$ there are at most $\gamma e_H(W_i, W_j)$ edges contained in the bipartite graphs $H[W_i^{(i')}, W_j^{(j')}]$, $i' \in [p_i], j' \in [p_j]$, with $\mathrm{d}_H(W_i^{(i')}, W_j^{(j')}) \leqslant \gamma^2$. Since $\sum_{ij \in E(F)} e_H(W_i, W_j) \leqslant kn$, the validity of (2.7) follows. Combining (2.5), (2.6), and (2.7) we finish the proof. □

3 Cutting trees: ℓ-fine partitions

The purpose of this section is to introduce some notation related to trees. The notion of an ℓ-fine partition of a tree shall be of particular interest. Roughly speaking, an ℓ-fine partition of a tree $T \in \mathbf{trees}(k)$ is a partition of the T into a small number of cut-vertices and subtrees of order at most ℓ with some additional properties. This notion is essential for our proof of Theorem 1.3 as we use a certain sequential procedure to embed $T_{\triangleright\mathrm{T}1.3}$ into the host graph $G_{\triangleright\mathrm{T}1.3}$, embedding a subtree after subtree.

Let T be a tree rooted at r, inducing the partial order $\preceq$ on $V(T)$ (with r as the minimal element). If $a \preceq b$ and $ab \in E(T)$ then we say b is a *child of* a and a is the *parent of* b. $\mathrm{Ch}(a)$

denotes the set of children of a, and the parent of a vertex $b \neq r$ is denoted $\mathrm{Par}(b)$. For a set $U \subseteq V(T)$ write $\mathrm{Par}(U) := \bigcup_{u \in U \setminus \{r\}} \mathrm{Par}(u) \setminus U$ and $\mathrm{Ch}(U) := \bigcup_{u \in U} \mathrm{Ch}(u) \setminus U$.

We say that a tree $T' \subseteq T$ is *induced* by a vertex $x \in V(T)$ if $V(T')$ is the up-closure of x in $V(T)$, i.e., $V(T') = \{v \in V(T) : x \preceq v\}$. We then write $T' = T(r, \uparrow x)$, or $T' = T(\uparrow x)$, if the root is obvious from the context and call T' an *end subtree*. Subtrees of T that are not end subtrees are called *internal subtrees*.

Let T be a tree rooted at r and let $T' \subseteq T$ be a subtree with $r \notin V(T')$. The *seed of* T' is the $\preceq$-maximal vertex $x \in V(T) \setminus V(T')$ such that $x \preceq v$ for all $v \in V(T')$. We write $\mathrm{Seed}(T') = x$. A *fruit* in a rooted tree (T, r) is any vertex $u \in V(T)$ whose distance from r is even and at least four.

We can now state the most important definition of this section.

Definition 3.1 (**ℓ-fine partition**). *Let $T \in$ **trees**(k) be a tree rooted at r. An ℓ-*fine partition of *T is a quadruple $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$, where $W_A, W_B \subseteq V(T)$ and $\mathcal{S}_A$, $\mathcal{S}_B$ are families of subtrees of T such that*

(a) the three sets W_A, W_B and $\{V(T^*)\}_{T^* \in \mathcal{S}_A \cup \mathcal{S}_B}$ partition $V(T)$,

(b) $r \in W_A \cup W_B$,

(c) $\max\{|W_A|, |W_B|\} \leqslant 336k/\ell$,

(d) for $w_1, w_2 \in W_A \cup W_B$ the distance $\mathrm{dist}(w_1, w_2)$ is odd if and only if one of them lies in W_A and the other one in W_B,

(e) $v(T^*) \leqslant \ell$ for every tree $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$,

(f) $V(T^*) \cap \mathrm{N}(W_B) = \emptyset$ for every $T^* \in \mathcal{S}_A$ and $V(T^*) \cap \mathrm{N}(W_A) = \emptyset$ for every $T^* \in \mathcal{S}_B$,

(g) each tree of $\mathcal{S}_A \cup \mathcal{S}_B$ has its seed in $W_A \cup W_B$,

(h) $|V(T^*) \cap \mathrm{N}(W_A \cup W_B)| \leqslant 2$ for each $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$,

(i) if $V(T^*) \cap \mathrm{N}(W_A \cup W_B)$ contains two distinct vertices y_1, y_2 for some $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$, then $\mathrm{dist}(y_1, y_2) \geqslant 4$,

(j) if $T_1, T_2 \in \mathcal{S}_A \cup \mathcal{S}_B$ are two internal subtrees of T such that $v_1 \in T_1$ precedes $v_2 \in T_2$ then $\mathrm{dist}_T(v_1, v_2) > 2$,

(k) $\mathcal{S}_B$ does not contain any internal tree of T, and

(l) $\sum_{\substack{T^* \in \mathcal{S}_A \\ T^* \text{ end tree of } T}} v(T^*) \geqslant \sum_{T^* \in \mathcal{S}_B} v(T^*)$.

Remark 3.2. *It is easy to see that any ℓ-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree (T, r) is determined once we know the set $W = W_A \cup W_B$, except possibly for being able to swap W_A with W_B and $\mathcal{S}_A$ with $\mathcal{S}_B$. Indeed, the division of W into two sets W' and W'' follows the bipartition of T, and conditions* (k) *and* (l) *determine which of W', W'' is W_A unless $T - W$ contains no internal trees and* (l) *would hold either way. During the proof of Lemma 3.4 below we shall therefore sometimes just say one of the conditions* (a)–(l) *holds for the set W, and not explicitly mention the tuple $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$.*

Remark 3.3. *Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ is an ℓ-fine partition of a tree (T, r), and suppose that $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$ is such that $|V(T^*) \cap \mathrm{N}(W_A \cup W_B)| = 2$. Let us root T^* at the neighbour r_1 of its seed, and let r_2 be the other vertex of $V(T^*) \cap \mathrm{N}(W_A \cup W_B)$. Then* (d), (f), *and* (i) *imply that r_2 is a fruit in (T^*, r_1).*

The following is the main lemma of this section. It asserts that each tree of order k has ℓ-fine partitions for all values of $\ell \leqslant k$.

Lemma 3.4. *Let $T \in \mathbf{trees}(k)$ be a tree rooted at r and let $\ell \in \mathbb{N}$ with $\ell \leqslant k$. Then T has an ℓ-fine partition.*

Similar but simpler tree-cutting procedures were used in other literature concerning the Loebl-Komlós-Sós Conjecture in the dense setting, cf. [AKS95, HP, PS12, Zha11]. There, using the notation of Conjecture 1.2, the trees in $\mathcal{S}_A \cup \mathcal{S}_B$ of an ℓ-fine partition of a tree $T \in \mathbf{trees}(k)$ are embedded in regular pairs of a Regularity Lemma decomposition of the host graph G. In the current paper however, a more complex decomposition result (Lemma 4.14) than the Regularity Lemma is used to capture the structure of G. To this end we had to further strengthen the features of the ℓ-fine partition. In particular, features (h), (i), (j) of Definition 3.1 were introduced to handle the more complex embedding procedures in our setting.

Remark 3.5.

(i) In our proof of Theorem 1.3, we shall apply Lemma 3.4 to a tree $T_{\rhd\mathrm{T}1.3} \in \mathbf{trees}(k)$. The number $\ell_{\rhd\mathrm{L}3.4}$ will be linear in k, and thus (c) *of Definition 3.1 tells us that the size of the sets W_A and W_B is bounded by an absolute constant.*

(ii) Each internal tree in $\mathcal{S}_A$ of an ℓ-fine partition has a unique vertex from W_A above it. Thus with $\ell_{\rhd\mathrm{L}3.4}$ as above also the number of internal trees in $\mathcal{S}_A$ is bounded by an absolute constant. This need not not be the case for the number of end trees. For instance, if $(T_{\rhd\mathrm{T}1.3}, r)$ is a star with $k-1$ leaves and rooted at its centre r then $W_A = \{r\}$ while the $k-1$ leaves of $T_{\rhd\mathrm{T}1.3}$ form the end shrubs in $\mathcal{S}_A$.

Proof of Lemma 3.4. First we shall use an inductive construction to get candidates for W_A, W_B, $\mathcal{S}_A$ and $\mathcal{S}_B$, which we shall modify later on, so that they satisfy all the conditions required by Definition 3.1.

Set $T_0 := T$. Now, inductively for $i \geqslant 1$ choose a $\preceq$-maximal vertex $x_i \in V(T_{i-1})$ with the property that $v(T_{i-1}(\uparrow x_i)) > \ell$. We set $T_i := T_{i-1} - (V(T_{i-1}(\uparrow x_i)) \setminus \{x_i\})$. If, say at step $i = i_{\mathrm{end}}$,

no such x_i exists, then $v(T_{i-1}) \leqslant \ell$. In that case, set $x_i := r$, set $W_1 := \{x_i\}_{i=1}^{i_{\text{end}}}$ and terminate. The fact that $v(T_{i-1} - V(T_i)) \geqslant \ell$ for each $i < i_{\text{end}}$ implies that

$$|W_1| - 1 = i_{\text{end}} - 1 \leqslant k/\ell \ . \tag{3.1}$$

Let $\mathcal{C}$ be the set of all components of the forest $T - W_1$. Observe that by the choice of the x_i each $T^* \in \mathcal{C}$ has order at most ℓ.

Let A and B be the colour classes of T such that $r \in A$. Now, choosing W_A as $W_1 \cap A$ and W_B as $W_1 \cap B$ and dividing $\mathcal{C}$ adequately into sets $\mathcal{S}_A$ and $\mathcal{S}_B$ would yield a quadruple that satisfies conditions (a), (b), (c), (d), (e) and (g). In order to find also the remaining properties satisfied, we shall refine our tree partition by adding more vertices to W_1, thus making the trees in $\mathcal{S}_A \cup \mathcal{S}_B$ smaller. In doing so, we have to be careful not to end up violating (c). We shall enlarge the set of cut vertices in several steps, accomplishing sequentially, in this order, also properties (h), (j), (f), (i), and in the last step at the same time (k) and (l). It will be easy to check that in each of the steps none of the previously established properties is lost, so we will not explicitly check them, except for (c).

For condition (h), first define T' as the subtree of T that contains all vertices of W_1 and all vertices that lie on paths in T which have both endvertices in W_1. Now, if a subtree $T^* \in \mathcal{C}$ does not already satisfy (h) for W_1, then $V(T^*) \cap V(T')$ must contain some vertices of degree at least three. We will add the set $Y(T^*)$ of all these vertices to W_1. Formally, let Y be the union of the sets $Y(T^*)$ over all $T^* \in \mathcal{C}$, and set $W_2 := W_1 \cup Y$. Then the components of $T - W_2$ satisfy (h).

Let us upper-bound the size of the set W_2. For each $T^* \in \mathcal{C}$, note that by Fact 2.2 for $T^* \cap T'$, we know that $|Y(T^*)|$ is at most the number of leaves of $T^* \cap T'$ (minus two). On the other hand, each leaf of $T^* \cap T'$ has a child in W_1 (in T). As these children are distinct for different trees $T^* \in \mathcal{C}$, we find that $|Y| \leqslant |W_1|$ and thus

$$|W_2| \leqslant 2|W_1| \ . \tag{3.2}$$

Next, for condition (j), observe that by setting $W_3 := W_2 \cup \mathrm{Par}_T(W_2)$ the components of $T - W_3$ fulfill (j). We have

$$|W_3| \leqslant 2|W_2| \overset{(3.2)}{\leqslant} 4|W_1| \ . \tag{3.3}$$

In order to ensure condition (f), let R^* be the set of the roots ($\preceq$-minimal vertices) of those components T^* of $T - W_3$ which contain neighbours of both colour classes of T. Setting $W_4 := W_3 \cup R^*$ we see that (f) is satisfied for W_4. Furthermore, as for each vertex in R^* there is a distinct member of W_3 above it in the order on T, we obtain

$$|W_4| \leqslant 2|W_3| \overset{(3.3)}{\leqslant} 8|W_1|. \tag{3.4}$$

Next, we shall aim for a stronger version of property (i), namely,

(i') if $V(T^*) \cap \mathrm{N}_T(W_A \cup W_B) = \{y_1, y_2\}$ with $y_1 \neq y_2$ for some $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$, then $\mathrm{dist}(y_1, y_2) \geqslant 6$.

The reason for requiring this strengthening is that later we might introduce additional cut vertices which would "shorten T^* by two".

Consider a component T^* of $T - W_4$ which is an internal tree of T. If T^* contains two distinct neighbours y_1, y_2 of W_4 such that $\mathrm{dist}_{T^*}(y_1, y_2) < 6$, then we call T^* *short*. Observe that there are at most $|W_4|$ short trees, because each of these trees has a unique vertex from W_4 above it. Let $Z(T^*) \subseteq V(T^*)$ be the vertices on the path from y_1 to y_2. Then $|Z(T^*)| \leqslant 6$. Letting Z be the union of the sets $Z(T^*)$ over all short trees in $T - W_4$, and set $W_5 := W_4 \cup Z$, we obtain

$$|W_5| \leqslant |W_4| + 6|W_4| \overset{(3.4)}{\leqslant} 56|W_1| \overset{(3.1)}{\leqslant} 112k/\ell. \tag{3.5}$$

We still need to ensure (k) and (l). To this end, consider the set $\mathcal{C}'$ of all components of $T - W_5$. Set $\mathcal{C}'_A := \{T^* \in \mathcal{C}' : \mathrm{Seed}(T^*) \in A\}$ and set $\mathcal{C}'_B := \mathcal{C}' \setminus \mathcal{C}'_A$. We assume that

$$\sum_{T^* \in \mathcal{C}'_A \,:\, T^* \text{ end tree of } T} v(T^*) \geqslant \sum_{T^* \in \mathcal{C}'_B \,:\, T^* \text{ end tree of } T} v(T^*)\ , \tag{3.6}$$

as otherwise we can simply swap A and B.

Now, for each $T^* \in \mathcal{C}'_B$ that is not a end subtree of T, set $X(T^*) := V(T^*) \cap \mathrm{N}_T(W_5)$. Let X be the union of all such sets $X(T^*)$. Observe that

$$|X| \leqslant 2|W_5 \cap B| \leqslant 2|W_5|. \tag{3.7}$$

For $W := W_5 \cup X$, all internal trees of $T - W$ have their seeds in A. This will guarantee (k), and, together with (3.6), also (l).

Finally, set $W_A := W \cap A$ and $W_B := W \cap B$, and let $\mathcal{S}_A$ and $\mathcal{S}_B$ be the sets of those components of $T - W$ that have their seeds in W_A and W_B, respectively. By construction, $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ has all the properties of an ℓ-fine partition. In particular, for (c), we find with (3.5) and (3.7) that $|W| \leqslant |W_5| + 2|W_5 \cap B| \leqslant 336k/\ell$. $\square$

For an ℓ-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a rooted tree (T, r), the trees $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$ are called *shrubs*. An *end shrub* is a shrub which is an end subtree. An *internal shrub* is a shrub which is an internal subtree. A *knag* is a component of the forest $T[W_A \cup W_B]$. Suppose that $T^* \in \mathcal{S}_A$ is an internal shrub, and r^* its $\preceq_r$-minimal vertex. Then $T^* - r^*$ contains a unique component with a vertex from $\mathrm{N}_T(W_A)$. We call this component *principal subshrub*, and the other components *peripheral subshrubs*.

Definition 3.6 (ordered skeleton). *We say that the sequence* $\big(X_0, X_1, \ldots, X_m\big)$ *is an* ordered skeleton *of the ℓ-fine partition* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *of a rooted tree* (T, r) *if*

- *X_0 is a knag and contains r, and all other X_i are either knags or shrubs,*
- *$V(\bigcup_{i \leqslant m} X_i) = V(T)$, and*
- *for each $i = 1, \ldots, m$, the subgraph formed by $X_0 \cup X_1 \cup \ldots \cup X_i$ is connected in T.*

Directly from Definition 3.1 we get:

Lemma 3.7. *Any ℓ-fine partition of any rooted tree has an ordered skeleton.*

Figure 3.1 shows an (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a binary tree $T \in \mathbf{trees}(k)$, for a fixed $\tau > 0$ and k large. The vertices whose distance is $O(\log(\tau^{-1}))$ from the root comprise a sole knag of T (with respect to $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$). This example will be important in Section 4.5.

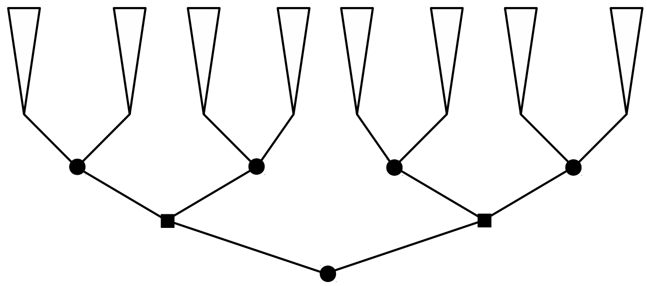

Figure 3.1: An (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a binary tree $T \in \mathbf{trees}(k)$. The elements of the set W_A are drawn as circles and those of W_B as squares. The sole knag is of depth $O(\log(\tau^{-1}))$, two in this picture. Each schematic triangle represents one end shrub of $\mathcal{S}_A \cup \mathcal{S}_B$.

4 Decomposing sparse graphs

In this section, we work out a structural decomposition of a possibly sparse graph which is suitable for embedding trees. Our motivation comes from the success of the Regularity Method in the setting of dense graphs (see [KO09]). The main technical result of this section, the "decomposition lemma", Lemma 4.13, provides such a decomposition. Roughly speaking, each graph of a moderate maximum degree can be decomposed into regular pairs, and two different expanding parts.

We then combine Lemma 4.13 with a lemma on creating a gap in the degree sequence (Lemma 4.1) to get a decomposition lemma for graphs from $\mathbf{LKS}(n, k, \eta)$, Lemma 4.14. Lemma 4.14 asserts that each graph from $\mathbf{LKS}(n, k, \eta)$ can be decomposed into vertices of degree much larger than k, regular pairs, and expanding parts. As a careful reader can check from the proof of Lemma 4.14 below, such a decomposition is possible for *any* graph; in Lemma 4.14 however we use properties specific to the class $\mathbf{LKS}(n, k, \eta)$ to get some additional features of the decomposition. Indeed, we expect that our technique will find applications in other tree embedding problems, and possibly elsewhere.

4.1 Creating a gap in the degree sequence

The goal of this section is to show that any graph $G \in \mathbf{LKSmin}(n, k, \eta)$ has a subgraph $G' \in \mathbf{LKSsmall}(n, k, \eta/2)$ which has a gap in its degree sequence. Note that G' then contains almost all the edges of G. This is formulated in the next lemma.

Lemma 4.1. *Let $G \in \mathbf{LKSmin}(n,k,\eta)$ and let $(\Omega_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers with $\Omega_j/\Omega_{j+1} \leqslant \eta^2/100$ for all $j \in \mathbb{N}$. Then there is an index $i^* \leqslant 100\eta^{-2}$ and a subgraph $G' \subseteq G$ such that*

(i) $G' \in \mathbf{LKSsmall}(n,k,\eta/2)$, and

(ii) no vertex $v \in V(G')$ has degree $\deg_{G'}(v) \in [\Omega_{i^*}k, \Omega_{i^*+1}k)$.

Proof. Set $R := \lfloor 100\eta^{-2}\rfloor$. For $i \in [R]$ and any graph $H \subseteq G$ define the sets $X_i(H) := \{v \in V(H) : \deg_H(v) \in [\Omega_i k, \Omega_{i+1}k)\}$ and for $i = R+1$ set $X_i(H) := \{v \in V(H) : \deg_H(v) \in [\Omega_i k, \infty)\}$. As

$$\sum_{i\in[R]} \quad \sum_{v\in X_i(G)\cup X_{i+1}(G)} \deg(v) \leqslant 4e(G) ,$$

by averaging we find an index $i^* \in [R]$ such that

$$\sum_{v\in X_{i^*}(G)\cup X_{i^*+1}(G)} \deg(v) \leqslant \frac{4e(G)}{R}. \tag{4.1}$$

Let E_0 be the set of all the edges incident with $X_{i^*}(G) \cup X_{i^*+1}(G)$. Now, starting with $G_0 := G - E_0$, successively define graphs $G_j \subsetneq G_{j-1}$ for $j \geqslant 1$ using any of the following two types of edge deletions:

(T1) If there is a vertex $v_j \in X_{i^*}(G_{j-1})$ then we choose an edge e_j that is incident with v_j, and set $G_j := G_{j-1} - e_j$.

(T2) If there is an edge $e_j = u_jv_j$ of G_{j-1} with $u_j \in \mathbb{S}_{\eta/2,k}(G_{j-1})$ and $v_j \in \bigcup_{i=i^*+1}^{R+1} X_i(G_{j-1})$ then we set $G_j := G_{j-1} - e_j$.

Since we keep deleting edges, the procedure stops at some point, say at step j^*, when neither of (T1), (T2) is applicable. Note that the resulting graph G_{j^*} already has Property (ii).

Let $E_1 \subseteq E(G)$ be the set of those edges deleted by applying (T1). We shall estimate the size of E_1. First, observe that

$$\left|\bigcup_{i=i^*+2}^{R+1} X_i(G)\right| \leqslant \frac{2e(G)}{\Omega_{i^*+2}k} .$$

Moreover, each vertex of $\bigcup_{i=i^*+2}^{R+1} X_i(G)$ appears at most $(\Omega_{i^*+1} - \Omega_{i^*})k < \Omega_{i^*+1}k$ times as the vertex v_j in the deletions of type (T1). Consequently,

$$|E_1| \leqslant \Omega_{i^*+1}\left|\bigcup_{i=i^*+2}^{R+1} X_i(G)\right| k \leqslant \frac{2\Omega_{i^*+1}e(G)}{\Omega_{i^*+2}} . \tag{4.2}$$

Now, observe that the vertices in $\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})$ have dropped their degree from $(1+\eta)k$ to $(1+\eta/2)k$ by operations other than (T2). So each of these vertices is incident with at least $\eta k/2$ edges from the set $E_0 \cup E_1$. Therefore, by the definition of E_0, by (4.1), and by (4.2),

$$\left|\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})\right| \leqslant \frac{2\cdot|E_0\cup E_1|}{\eta k/2} \leqslant \left(\frac{4}{R} + \frac{2\Omega_{i^*+1}}{\Omega_{i^*+2}}\right)\cdot\frac{4e(G)}{\eta k} \overset{(2.1)}{\leqslant} \frac{\eta n}{2} .$$

Thus

$$|\mathbb{L}_{\eta/2,k}(G_{j^*})| \geqslant |\mathbb{L}_{\eta,k}(G)| - |\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})| \geqslant (1/2+\eta/2)n \ ,$$

and consequently, $G_{j^*} \in \mathbf{LKS}(n,k,\eta/2)$.

Last, we obtain the graph G' by successively deleting any edge from G_{j^*} which connects a vertex from $\mathbb{S}_{\eta/2,k}(G_{j^*})$ with a vertex whose degree is not exactly $\lceil(1+\frac{\eta}{2})k\rceil$. This does not affect the already obtained Property (ii), since we could not apply (T2) to G_{j^*}. We claim that for the resulting graph G' we have $G' \in \mathbf{LKSsmall}(n,k,\eta/2)$. Indeed, $\mathbb{L}_{\eta/2,k}(G') = \mathbb{L}_{\eta/2,k}(G_{j^*})$, and thus $G' \in \mathbf{LKS}(n,k,\eta/2)$. Property 2 of Definition 2.6 follows from the last step of the construction of G'. To see Property 1 of Definition 2.6 we use Fact 2.5(2) for G (which by assumption is in $\mathbf{LKSmin}(n,k,\eta)$). □

4.2 Decomposition of graphs with moderate maximum degree

First we introduce some useful notions. We start with dense spots which indicate an accumulation of edges in a sparse graph.

Definition 4.2 (**(m,γ)-dense spot, (m,γ)-nowhere-dense**)**.** *An (m,γ)-*dense spot *in a graph G is a non-empty bipartite subgraph $D=(U,W;F)$ of G with $\mathrm{d}(D)>\gamma$ and $\mathrm{deg}^{\min}(D)>m$. We call G (m,γ)-*nowhere-dense *if it does not contain any (m,γ)-dense spot.*

We remark that dense spots as bipartite graphs do not have a specified orientation, that is, we view $(U,W;F)$ and $(W,U;F)$ as the same object.

Fact 4.3. *Let $(U,W;F)$ be a $(\gamma k,\gamma)$-dense spot in a graph G of maximum degree at most Ωk. Then* $\max\{|U|,|W|\} \leqslant \frac{\Omega}{\gamma}k$.

Proof. It suffices to observe that

$$\gamma|U||W| \leqslant e(U,W) \leqslant \mathrm{deg}^{\max}(G)\cdot\min\{|U|,|W|\} \leqslant \Omega k\cdot\min\{|U|,|W|\}.$$

□

The next fact asserts that in a bounded degree graph there cannot be too many edge-disjoint dense spots containing a given vertex.

Fact 4.4. *Let H be a graph of maximum degree at most Ωk, let $v\in V(H)$, and let $\mathcal{D}$ be a family of edge-disjoint $(\gamma k,\gamma)$-dense spots. Then less than $\frac{\Omega}{\gamma}$ dense spots from $\mathcal{D}$ contain v.*

Proof. This follows as v sends more than γk edges to each dense spot from $\mathcal{D}$ it is incident with, the dense spots $\mathcal{D}$ are edge-disjoint, and $\deg(v)\leqslant\Omega k$. □

Last, we include a bound concerning the total size of dense spots intersecting substantially a given set.

Fact 4.5. *Let H be a graph of maximum degree at most Ωk. Let $Y \subseteq V(H)$ be a set of size at most Ak, and $\mathcal{D}$ a family of edge-disjoint $(\gamma k, \gamma)$-dense spots. Define $\mathcal{D}' := \{D \in \mathcal{D} : |V(D) \cap Y| \geqslant \beta k\}$. Then for the set $X := \bigcup_{D \in \mathcal{D}'} V(D)$ we have $|X| \leqslant \frac{2A\Omega^2}{\beta\gamma^2} k$.*

Proof. Let us count the number of certain pairs (y, D) in two different ways.

$$\beta k |\mathcal{D}'| \leqslant \left| \{(y, D) \ : \ y \in Y, D \in \mathcal{D}', y \in V(\mathcal{D}')\} \right| \overset{\text{F4.4}}{\leqslant} |Y| \frac{\Omega}{\gamma} \, .$$

Put together, $|\mathcal{D}'| \leqslant \frac{A\Omega}{\beta\gamma}$. The fact now follows from Fact 4.3. $\square$

Our second definition of this section might seem less intuitive at first sight. It describes a property for finding dense spots outside some "forbidden" set U, which in later applications will be the set of vertices already used for a partial embedding of a tree $T_{\triangleright\text{T1.3}} \in \mathbf{trees}(k)$ in Theorem 1.3 during our sequential embedding procedure.

Definition 4.6 (**$(\Lambda, \varepsilon, \gamma, k)$-avoiding set**)**.** *Suppose that G is a graph and $\mathcal{D}$ is a family of dense spots in G. A set $\mathfrak{A} \subseteq \bigcup_{D \in \mathcal{D}} V(D)$ is* $(\Lambda, \varepsilon, \gamma, k)$-avoiding *with respect to $\mathcal{D}$ if for every $\bar{U} \subseteq V(G)$ with $|\bar{U}| \leqslant \Lambda k$ the following holds that for all but at most εk vertices $v \in \mathfrak{A}$. There is a dense spot $D \in \mathcal{D}$ with $|\bar{U} \cap V(D)| \leqslant \gamma^2 k$ that contains v.*

Note that a subset of a $(\Lambda, \varepsilon, \gamma, k)$-avoiding set is also $(\Lambda, \varepsilon, \gamma, k)$-avoiding.

We now come to the main concepts of this section, the bounded and the sparse decompositions. These notions in a way correspond to the partition structure from the Regularity Lemma, although naturally more complex since we deal with (possibly) sparse graphs here. Lemma 4.13 is then a corresponding regularization result.

Definition 4.7 (**$(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition**)**.** *Let $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ be a partition of the vertex set of a graph G. We say that $(\mathbf{V}, \mathcal{D}, G_{\text{reg}}, G_{\text{exp}}, \mathfrak{A})$ is a* $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition *of G with respect to $\mathcal{V}$ if the following properties are satisfied:*

1. *The elements of $\mathbf{V}$ are disjoint subsets of $V(G)$.*

2. *G_{reg} is a subgraph of $G - G_{\text{exp}}$ on the vertex set $\bigcup \mathbf{V}$. For each edge $xy \in E(G_{\text{reg}})$ there are distinct $C_x \ni x$ and $C_y \ni y$ from $\mathbf{V}$, and $G[C_x, C_y] = G_{\text{reg}}[C_x, C_y]$. Furthermore, $G[C_x, C_y]$ forms an ε-regular pair of density at least γ^2.*

3. *We have $\nu k \leqslant |C| = |C'| \leqslant \varepsilon k$ for all $C, C' \in \mathbf{V}$.*

4. *$\mathcal{D}$ is a family of edge-disjoint $(\gamma k, \gamma)$-dense spots in $G - G_{\text{exp}}$. For each $D = (U, W; F) \in \mathcal{D}$ all the edges of $G[U, W]$ are covered by $\mathcal{D}$ (but not necessarily by D).*

5. *If G_{reg} contains at least one edge between $C_1, C_2 \in \mathbf{V}$ then there exists a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $C_1 \subseteq U$ and $C_2 \subseteq W$.*

6. *For all $C \in \mathbf{V}$ there is $V \in \mathcal{V}$ so that either $C \subseteq V \cap V(G_{\rm exp})$ or $C \subseteq V \setminus V(G_{\rm exp})$. For all $C \in \mathbf{V}$ and $D = (U, W; F) \in \mathcal{D}$ we have $C \cap U \in \{\emptyset, C\}$.*

7. *$G_{\rm exp}$ is a $(\gamma k, \gamma)$-nowhere-dense subgraph of G with $\mathrm{deg}^{\min}(G_{\rm exp}) > \rho k$.*

8. *$\mathfrak{A}$ is a $(\Lambda, \varepsilon, \gamma, k)$-avoiding subset of $V(G) \setminus \bigcup \mathbf{V}$ with respect to dense spots $\mathcal{D}$.*

We say that the bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\rm reg}, G_{\rm exp}, \mathfrak{A})$ respects the avoiding threshold b *if for each $C \in \mathbf{V}$ we either have $\mathrm{deg^{max}}_G(C, \mathfrak{A}) \leqslant b$, or $\mathrm{deg^{min}}_G(C, \mathfrak{A}) > b$.*

Let us remark that "exp" in $G_{\rm exp}$ stands for "expander" and "reg" in $G_{\rm reg}$ stands for "regular(ity)".

The members of $\mathbf{V}$ are called *clusters*. Define the *cluster graph* $\mathbf{G}_{\rm reg}$ as the graph on the vertex set $\mathbf{V}$ that has an edge C_1C_2 for each pair (C_1, C_2) which has density at least γ^2 in the graph $G_{\rm reg}$.

Property 6 tells us that the clusters may be prepartitioned, just as it is the case in the classic Regularity Lemma. When classifying the graph $G_{\triangleright\mathrm{T}1.3}$ in Lemma 4.14 below we shall use the prepartition into (roughly) $\mathbb{S}_{\alpha_{\triangleright\mathrm{T}1.3},k}(G_{\triangleright\mathrm{T}1.3})$ and $\mathbb{L}_{\alpha_{\triangleright\mathrm{T}1.3},k}(G_{\triangleright\mathrm{T}1.3})$.

As said above, the notion of bounded decomposition is needed for our Regularity Lemma type decomposition given in Lemma 4.13. It turns out that such a decomposition is possible only when the graph is of moderate maximum degree. On the other hand, Lemma 4.1 tells us that the vertex set of any graph[5] can be decomposed into vertices of enormous degree and moderate degree. The graph induced by the latter type of vertices then admits the decomposition from Lemma 4.13. Thus, it makes sense to enhance the structure of bounded decomposition by vertices of unbounded degree. This is done in the next definition.

Definition 4.8 (**$(k, \Omega^{**}, \Omega^{*}, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition).** *Let $\mathcal{V} = \{V_1, V_2, \dots, V_s\}$ be a partition of the vertex set of a graph G. We say that $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\rm reg}, G_{\rm exp}, \mathfrak{A})$ is a* $(k, \Omega^{**}, \Omega^{*}, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition *of G with respect to $V_1, V_2, \dots, V_s$ if the following holds.*

1. *$\mathbf{\Psi} \subseteq V(G)$, $\mathrm{deg^{min}}_G(\mathbf{\Psi}) \geqslant \Omega^{**}k$, $\mathrm{deg^{max}}_H(V(G) \setminus \mathbf{\Psi}) \leqslant \Omega^{*}k$, where H is spanned by the edges of $\bigcup \mathcal{D}$, $G_{\rm exp}$, and edges incident with $\mathbf{\Psi}$,*

2. *$(\mathbf{V}, \mathcal{D}, G_{\rm reg}, G_{\rm exp}, \mathfrak{A})$ is a $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition of $G - \mathbf{\Psi}$ with respect to $V_1 \setminus \mathbf{\Psi}, V_2 \setminus \mathbf{\Psi}, \dots, V_s \setminus \mathbf{\Psi}$.*

If the parameters do not matter, we call ∇ simply a *sparse decomposition*, and similarly we speak about a *bounded decomposition*.

Definition 4.9 (**captured edges).** *In the situation of Definition 4.8, we refer to the edges in $E(G_{\rm reg}) \cup E(G_{\rm exp}) \cup E_G(\mathbf{\Psi}, V(G)) \cup E_G(\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V})$ as* captured *by the sparse decomposition.*

[5]Lemma 4.1 is stated only for graphs from $\mathbf{LKSmin}(n, k, \eta)$, but a similar statement can be made about any graph.

We write G_∇ for the subgraph of G on the same vertex set which consists of the captured edges. Likewise, the captured edges of a bounded decomposition $(\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$ of a graph G are those in $E(G_{\mathrm{reg}})\cup E(G_{\mathrm{exp}})\cup E_G(\mathfrak{A},\mathfrak{A}\cup\bigcup\mathbf{V})$.

Throughout the paper we write $G_\mathcal{D}$ for the subgraph of G which consists of the edges contained in $\mathcal{D}$. We now include an easy fact about the relation of $G_\mathcal{D}$ and G_{reg}.

Fact 4.10. *Let $\nabla=(\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$ be a sparse decomposition of a graph G. Then each edge $xy\in E(G_\mathcal{D})$ with $x,y\in\bigcup\mathbf{V}$ is either contained in G_{reg}, or is not captured.*

Proof. Indeed, suppose that $xy\in E(G_\mathcal{D})$, $x,y\in\bigcup\mathbf{V}$, and $xy\notin E(G_{\mathrm{reg}})$. Property 2 of Definition 4.8 says that $x,y\notin\mathbf{\Psi}$. Further, by Property 8 of Definition 4.7, we have $x,y\notin\mathfrak{A}$. Last, Property 4 of Definition 4.7 implies that $xy\notin E(G_{\mathrm{exp}})$. Hence xy is not captured, as desired. □

We now give a bound on the number of clusters reachable through edges of the dense spots from a fixed vertex outside $\mathbf{\Psi}$.

Fact 4.11. *Let $\nabla=(\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$ be a $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition of a graph G. Let $x\in V(G)\setminus\mathbf{\Psi}$. Assume that $\mathbf{V}\neq\emptyset$, and let $\mathfrak{c}$ be the size of each of the members of $\mathbf{V}$. Then there are less than*

$$\frac{2(\Omega^*)^2k}{\gamma^2\mathfrak{c}}\leqslant\frac{2(\Omega^*)^2}{\gamma^2\nu}$$

clusters $C\in\mathbf{V}$ with $\deg_{G_\mathcal{D}}(x,C)>0$.

Proof. Property 1 of Definition 4.8 says that $\deg_{G_\mathcal{D}}(x)\leqslant\Omega^*k$. For each $D\in\mathcal{D}$ with $x\in V(D)$ we have that $\deg_D(x)>\gamma k$, since D is a $(\gamma k,\gamma)$-dense spot. By Fact 4.4

$$|\{D\in\mathcal{D}:\deg_D(x)>0\}|<\frac{\Omega^*}{\gamma}. \tag{4.3}$$

Furthermore, by Fact 4.3, and using Property 3 of Definition 4.7, we see that for a fixed $D\in\mathcal{D}$, we have

$$|\{C\in\mathbf{V}\ :\ C\subseteq V(D)\}|\leqslant\frac{2\Omega^*k}{\gamma}\cdot\frac{1}{\mathfrak{c}}\leqslant\frac{2\Omega^*}{\gamma\nu}\ .$$

Together with (4.3) this gives that the number of clusters $C\in\mathbf{V}$ with $\deg_{G_\mathcal{D}}(x,C)>0$ is less than

$$\frac{\Omega^*}{\gamma}\cdot\frac{2\Omega^*k}{\gamma\mathfrak{c}}\leqslant\frac{\Omega^*}{\gamma}\cdot\frac{2\Omega^*}{\gamma\nu}\ ,$$

as desired. □

As a last step before we state the main result of this section we show that the cluster graph $\mathbf{G}_{\mathrm{reg}}$ corresponding to a $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition $(\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$ has bounded degree.

Fact 4.12. *Let $\nabla=(\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$ be a $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition of a graph G, and let $\mathbf{G}_{\mathrm{reg}}$ be the corresponding cluster graph. Let $\mathfrak{c}$ be the size of each cluster in $\mathbf{V}$. Then $\deg^{\max}(\mathbf{G}_{\mathrm{reg}})\leqslant\frac{\Omega^*k}{\gamma^2\mathfrak{c}}\leqslant\frac{\Omega^*}{\gamma^2\nu}$.*

Proof. Let $C \in \mathbf{V}$. Then by the definition of $\mathbf{G}_{\mathrm{reg}}$, and by the properties of Definitions 4.7 and 4.8, we get

$$\deg_{\mathbf{G}_{\mathrm{reg}}}(C) \leqslant \sum_{C' \in \mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(C)} \frac{e_{G_{\mathrm{reg}}}(C,C')}{\gamma^2|C||C'|} \leqslant \frac{\Omega^* k|C|}{\gamma^2|C|\mathfrak{c}} \leqslant \frac{\Omega^*}{\gamma^2\nu},$$

as desired. □

We now state the most important lemma of this section. It says that any graph of bounded degree has a bounded decomposition which captures almost all its edges. This lemma can be considered as a sort of Regularity Lemma for sparse graphs.

Lemma 4.13 (Decomposition lemma)**.** *For each $\Lambda, \Omega, s \in \mathbb{N}$ and each $\gamma, \varepsilon, \rho > 0$ there exist $k_0 \in \mathbb{N}$, $\nu > 0$ such that for every $k \geqslant k_0$ and every n-vertex graph G with $e(G) \leqslant kn$, $\deg^{\max}(G) \leqslant \Omega k$, and with a given partition $\mathcal{V}$ of its vertex set into at most s sets, there exists a $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ with respect to $\mathcal{V}$, which captures all but at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma + \rho)kn$ edges of G. Furthermore, this bounded decomposition respects any given avoiding threshold b and we have*

$$|E(\mathcal{D}) \setminus (E(G_{\mathrm{reg}}) \cup E_G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])| \leqslant (\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma)kn\ . \tag{4.4}$$

A proof of Lemma 4.13 is given in Section 4.6.

4.3 Decomposition of LKS graphs

Lemma 4.1 and Lemma 4.13 enable us to decompose graphs in $\mathbf{LKS}(n, k, \eta)$ in a particular manner.

Lemma 4.14. *For every $\eta, \Lambda, \gamma, \varepsilon, \rho > 0$ there are $\nu > 0$ and $k_0 \in \mathbb{N}$ such that for every $k > k_0$ and for every number b the following holds. For every sequence $(\Omega_j)_{j\in\mathbb{N}}$ of positive numbers with $\Omega_j/\Omega_{j+1} \leqslant \eta^2/100$ for all $j \in \mathbb{N}$ and for every $G \in \mathbf{LKS}(n, k, \eta)$ there are an index i and a subgraph G' of G with the following properties:*

(a) $G' \in \mathbf{LKSsmall}(n, k, \eta/2)$,

(b) $i \leqslant 100\eta^{-2}$,

(c) *G' has a $(k, \Omega_{i+1}, \Omega_i, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G'_{\mathrm{reg}}, G'_{\mathrm{exp}}, \mathfrak{A})$ with respect to the partition $\{V_1, V_2\} := \{\mathbb{S}_{\eta/2,k}(G'), \mathbb{L}_{\eta/2,k}(G')\}$, and with respect to avoiding threshold b,*

(d) *$(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G'_{\mathrm{reg}}, G'_{\mathrm{exp}}, \mathfrak{A})$ captures all but at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega_{\lfloor 100\eta^{-2}\rfloor} + \gamma + \rho)kn$ edges of G', and*

(e) $|E(\mathcal{D}) \setminus (E(G'_{\mathrm{reg}}) \cup E_{G'}[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])| \leqslant (\frac{4\varepsilon}{\gamma} + \varepsilon\Omega_{\lfloor 100\eta^{-2}\rfloor} + \gamma)kn$.

Proof. Let ν and k_0 be given by Lemma 4.13 for input parameters $\Omega_{\triangleright\mathrm{L}4.13} := \Omega_{\lfloor 100\eta^{-2}\rfloor}$, $\Lambda_{\triangleright\mathrm{L}4.13} := \Lambda$, $\gamma_{\triangleright\mathrm{L}4.13} := \gamma$, $\varepsilon_{\triangleright\mathrm{L}4.13} := \varepsilon$, $\rho_{\triangleright\mathrm{L}4.13} := \rho$, $b_{\triangleright\mathrm{L}4.13} := b$, and $s_{\triangleright\mathrm{L}4.13} := 2$. Now, given G, let us consider a subgraph $\tilde{G}$ of G such that $\tilde{G} \in \mathbf{LKSmin}(n, k, \eta)$. Lemma 4.1 applied to the

sequence $(\Omega_j)_j$ and $\tilde{G}$ yields a graph $G' \in \mathbf{LKSsmall}(n,k,\eta/2)$ and an index $i \leqslant 100\eta^{-2}$. We set $\mathbf{\Psi} := \{v \in V(G) \,:\, \deg_{G'}(v) \geqslant \Omega_{i+1}k\}$.

Observe that by (2.1), $e(G') < kn$. Let $(\mathbf{\Psi}, \mathcal{D}, G'_{\mathrm{reg}}, G'_{\mathrm{exp}}, \mathfrak{A})$ be the $(k,\Lambda,\gamma,\varepsilon,\nu,\rho)$-bounded decomposition of the graph $G' - \mathbf{\Psi}$ with respect to $\{\mathbb{S}_{\eta/2,k}(G'), \mathbb{L}_{\eta/2,k}(G') \setminus \mathbf{\Psi}\}$ that is given by Lemma 4.13. Clearly, $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G'_{\mathrm{reg}}, G'_{\mathrm{exp}}, \mathfrak{A})$ is a $(k,\Omega_{i+1},\Omega_i,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition of G' capturing at least as many edges as promised in the statement of the lemma. $\square$

A version of Lemma 4.14 could be formulated for a general n-vertex graph with $\Theta(kn)$ edges. It would assert that such a graph has a sparse classification which captures all but at most $o(kn)$ edges. Such a lemma could be used to attack other problems. However, our feeling is that such a decomposition lemma is limited in applications to tree-containment problems. The reason is that two of the features of the sparse decomposition, the nowhere-dense graph G_{exp} and the avoiding set $\mathfrak{A}$, seem to be useful only for embedding trees. See Section 4.4 and Section 4.5 for a discussion of the respective embedding strategies.

The process of embedding a given tree $T_{\triangleright\mathrm{T}1.3} \in \mathbf{trees}(k)$ into $G_{\triangleright\mathrm{T}1.3}$ is based on the sparse decomposition $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of a graph G from Lemma 4.14 and is much more complex than in approaches based on the standard Regularity Lemma. The embedding ingredient in the classic (dense) Regularity Method inheres in Blow-up Lemma type statements which roughly tell that regular pairs of positive density in some sense behave like complete bipartite graphs. In our setting, in addition to regular pairs[6] we shall use three other components of ∇: the vertices of huge degree $\mathbf{\Psi}$, the nowhere-dense graph G_{exp}, and the avoiding set $\mathfrak{A}$. Each of these components requires a different strategy for embedding (parts of) $T_{\triangleright\mathrm{T}1.3}$. Let us mention that rather major technicalities arise when combining these strategies; for example, for traversing between $\mathbf{\Psi}$ and the rest of the graph we have to introduce a certain "cleaned" structure in Lemma 7.33.

These strategies are described precisely and in detail in Section 8. A lighter informal account on the role of $\mathfrak{A}$ is given in Section 4.4. We discuss the use of G_{exp} in Section 4.5. Only very little can be said about the set $\mathbf{\Psi}$ at an intuitive level: these vertices have huge degrees but are very unstructured otherwise. If only $o(kn)$ edges are incident with $\mathbf{\Psi}$ then we can neglect them. If, on the other hand, there are $\Omega(kn)$ edges incident with $\mathbf{\Psi}$, then we have no choice but to use them for our embedding. Very roughly speaking, in that case we find sets $\mathbf{\Psi}' \subseteq \mathbf{\Psi}$ and $V' \subseteq V(G) \setminus \mathbf{\Psi}$ such that still $\deg^{\min}(\mathbf{\Psi}', V') \gg k$, and $\deg^{\min}(V', \mathbf{\Psi}') = \Omega(k)$, and then use $\mathbf{\Psi}'$ and V' in our embedding.

Last, let us note that when $G_{\triangleright\mathrm{T}1.3}$ is close to the extremal graph (depicted in Figure 1.1) then all the structure in $G_{\triangleright\mathrm{T}1.3}$ captured by Lemma 4.14 accumulates in the cluster graph G'_{reg}, i.e., $\mathbf{\Psi}$, G'_{exp} and $\mathfrak{A}$ are all almost empty. For that reason, when some of $\mathbf{\Psi}$, G'_{exp} or $\mathfrak{A}$ is substantial we gain some extra aid. In comparison, one of the almost extremal graphs for the Erdős-Sós Conjecture 1.1 has a substantial $\mathbf{\Psi}$-component (see Figure 1.2).

[6]Some of the regular pairs we shall use are already present in G_{reg}, and there are some additional regular pairs hidden in $\mathcal{D}$ which we shall extract and make use of in a form of so-called semiregular matchings (Definition 5.4) in Sections 5 and 6.

4.4 The role of the avoiding set $\mathfrak{A}$

Let us explain the role of the avoiding set $\mathfrak{A}$ in Lemma 4.13. As said above, our aim in Lemma 4.13 will be to locally regularize parts of the input graph G. Of course, first we try to regularize as large a part of the G as possible. The avoiding set arises as a result of the impossibility to regularize certain parts of the graph. Indeed, it is one of the most surprising steps in our proof of Theorem 1.3 that the set $\mathfrak{A}$ is initially defined as – very loosely speaking – "those vertices where the Regularity Lemma fails to work properly", and only then we prove[7] that $\mathfrak{A}$ actually satisfies the useful conditions of Definition 4.6.

We now sketch how to utilize avoiding sets for the purpose of embedding trees. In our proof of Theorem 1.3 we preprocess the tree $T = T_{\triangleright\mathrm{T}1.3} \in \mathbf{trees}(k)$ by considering its (τk)-fine partition, and then sequentially embed its shrubs (and knags). Thus embedding techniques for embedding a single shrub are the building blocks of our embedding machinery; and $\mathfrak{A}$ is one of the enviroments which provides us with such a technique. Let us discuss here the simpler case of end shrubs. More precisely, we show how to extend a partial embedding of a tree by one end-shrub. To this end, let us suppose that ϕ is a partial embedding of a tree T, and $v \in V(T)$ is its *active vertex*, i.e., a vertex which is embedded, but not all its children are. We write $U \subseteq V(G)$ for the current image of ϕ. Let $T' \subseteq T$ be an end-shrub which is not embedded yet, and suppose $u \in V(T')$ is adjacent to v. We have $v(T') \leqslant \tau k$.

We now show how to extend the partial embedding ϕ to T', assuming that $\deg_G\big(\phi(v), \mathfrak{A} \setminus U\big) \geqslant \gamma k$ for some $(1, \varepsilon, \gamma, k)$-avoiding set $\mathfrak{A}$ (where $\tau \ll \varepsilon \ll \gamma \ll 1$). Let X be the set of at most εk exceptional vertices from Definition 4.6 corresponding to the set U. We now embed T' into G, starting by embedding u in a vertex of $\mathfrak{A} \setminus (U \cup X)$ in the neighborhood of $\phi(v)$. By Definition 4.6, there is a dense spot $D = (A_D, B_D; F) \in \mathcal{D}$ such that $\phi(u) \in V(D)$ and $|U \cap V(D)| \leqslant \gamma^2 k$. As D is a dense spot, we have $\deg_G(\phi(u), V(D)) > \gamma k$. It is now easy to embed T' into D using the minimum degree in D. See Figure 4.1 for an illustration, and Lemma 8.3 for a precise formulation.

We indeed use the avoiding set for embedding shrubs of a fine partition of T as above. The major simplification we made in the exposition is that we only discussed the case when T' is an end shrub. To cover embedding of an internal shrub T' as well, one needs to have a more detailed control over the embedding, i.e., one must be able to extend the embedding from leaves of T' to the neighboring cut-vertices of the fine partition, is such a way that one can then continue embedding of the shrubs below these cut-vertices.

Last, let us remark, that unlike our baby-example above, we use an $(\Lambda, \varepsilon, \gamma, k)$-avoiding set with $\Lambda \gg 1$. This is because in the actual proof one has to avoid more vertices than just the current image of the embedding.

[7]See the last step of the proof of Lemma 4.13.

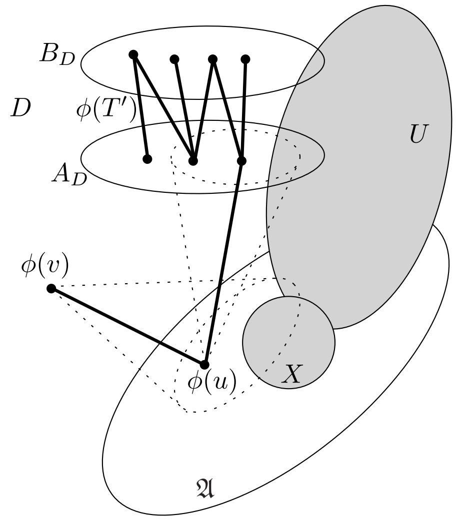

Figure 4.1: Embedding using the set $\mathfrak{A}$.

4.5 The role of the nowhere-dense graph G_{exp} and using the (τk)-fine partition

In this section we shall give some intuition on how the $(\gamma k, \gamma)$-nowhere-dense graph G_{exp} from the $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition[8] $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of a graph G is useful for embedding a given tree $T \in \mathbf{trees}(k)$. We start out with the rather simple case when T is a path. We then point out an issue with this approach for trees with many branching vertices and show how to overcome this problem using the (τk)-fine partition from Lemma 3.4.

Embedding a path in G_{exp}. Assume we are given a path $T = u_1 u_2 \cdots u_k \in \mathbf{trees}(k)$ and we wish to embed it into G_{exp}. The naive idea is to apply a one-step look-ahead strategy. We first embed u_1 in an arbitrary vertex $v \in V(G_{\mathrm{exp}})$. Then, we extend our embedding ϕ_ℓ of the path $u_1 \cdots u_\ell$ in G_{exp} in step ℓ by embedding $u_{\ell+1}$ in a (yet unused) neighbour w of the image of the *active* vertex u_ℓ, requiring that

$$\deg_{G_{\mathrm{exp}}}\big(w, \phi_\ell(u_1 \cdots u_\ell)\big) < \sqrt{\gamma} k \ . \tag{4.5}$$

Let us argue that such a vertex w exists. First, observe that Property 7 of Definition 4.7 implies that $\phi_\ell(u_\ell)$ has at least ρk neighbours. By (4.5) applied to $\ell - 1$, at most $\sqrt{\gamma} k$ of these neighbours lie inside $\phi_\ell(u_1 \cdots u_{\ell-1})$; this property is also trivially satisfied when $\ell = 1$. Further, an easy calculation shows that at most $16\sqrt{\gamma} k$ of them have degree more than $\sqrt{\gamma} k$ in G_{exp} into the set $\phi_\ell(u_1 \cdots u_\ell)$, otherwise we would get a contradiction to G_{exp} being $(\gamma k, \gamma)$-nowhere-dense. Since we assumed $\rho > 17\sqrt{\gamma}$ we can find a vertex w as desired and thus embed all of T.

[8] We shall assume that $17\sqrt{\gamma} < \rho$; this will be the setting of the sparse decomposition we shall work with in the proof of Theorem 1.3.

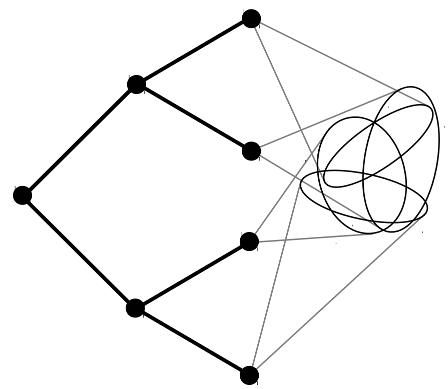

Figure 4.2: Embedded part of the binary tree in bold. The neighbourhoods of active vertices may overlap.

Embedding trees with many branching points and the role of fine partitions. We certainly cannot hope that a nonempty graph $G_{\exp}$ alone will provide us with embeddings of all trees $T \in \mathbf{trees}(k)$ from Theorem 1.3. For instance, if T is a star, then we need in G a vertex of degree $k-1$, which $G_{\exp}$ might not have. In order to run into a problem with the method described above, we do not even need to have such a large degree in our tree T.

Consider a binary tree $T \in \mathbf{trees}(k)$, rooted at its central vertex r. Now if we try to embed T sequentially as above we will arrive at a moment when there are many (as many as $k/2$) active vertices; regardless in which order we embed. Now, the neighbourhoods of the images of the active vertices cannot be controlled much, i.e., they may be intersecting considerably. Hence, embedding children of active vertices we might block available space in the neighbourhoods of other active vertices. See Figure 4.2 for an illustration.

To rescue the situation we use the (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T (for some $0 < \tau \ll \gamma$) given by Lemma 3.4. Recall the structure of this partition, as shown in Figure 3.1: the first q levels of T from the root r comprise the sole knag. All other vertices make up the end shrubs $T_1^*, \ldots, T_h^*$.

We first embed the knag, which consists of the cut vertices $W_A \cup W_B$, and so has size at most $O(\frac{1}{\tau})$. As ρk will be much larger than that, following a strategy similar to the one above we ensure that all of $W_A \cup W_B$ gets correctly embedded, we even have a (limited) choice for its images. The next step is to make the transitions at the q-th level from embedding cut vertices $W_A \cup W_B$ to embedding shrubs $T_1^*, \ldots, T_h^*$. But since this step requires to exploit the structure of LKS graphs, we skip the details in the high-level overview here. We just remark that one needs to put the cut vertices $W_A \cup W_B$ in the sets $\mathbb{XA}$ and $\mathbb{XB}$ from Lemma 6.1; these vertices are powerful enough to allow such a transition.

For the point we wish to make here, it is more relevant to see how to complete the last part of our embedding, that is, how to embed a tree T_i^* whose root r_i is already embedded in a vertex $\phi(r_i) \in V(G_{\exp})$. Let $\mathrm{im}_i := \mathrm{im}(\phi)$ be the current (partial) image of ϕ at this stage. We emphasize that at this moment we are working exclusively with the tree T_i^*, i.e., any other tree T_j^* is either

completely embedded, or will be embedded only after we finish the embedding of T_i^*. Suppose we are about to embed a vertex $v \in V(T_i^*)$ whose ancestor $v' \in V(T_i^*)$ is already embedded in $V(G_{\mathrm{exp}})$. We choose for the image of v any (yet unused) vertex w in the neighbourhood of $\varphi(v')$, requiring that

$$\deg_{G_{\mathrm{exp}}}(w, \mathrm{im}_i) < \rho k/100. \tag{4.6}$$

This condition is very similar to our path-embedding procedure above, and can be proved in exactly the same way, using the fact that G_{exp} is $(\gamma k, \gamma)$-nowhere-dense. Note that during our embedding $|\mathrm{im}(\phi) \setminus \mathrm{im}_i|$ will grow, but however is at most $v(T_i^*) \leqslant \tau k$. Thus, for every vertex $v'' \in V(T_i^*)$, when its time comes to be embedded, we still have $\deg_{G_{\mathrm{exp}}}\big(\phi(v'), \mathrm{im}(\phi)\big) \leqslant \rho k/100 + \tau k < \rho k/99$, and thus v'' can be embedded.

Note that the trick here was to keep on working on one subtree T_i^*, whose size is small enough to be negligible in comparison to the degree of a vertex in G_{exp} so that it does not matter that the set we wish to avoid having a considerable degree into $(\mathrm{im}(\phi))$ is not the same as the one we can actually avoid having a considerable degree into (im_i). (Observe that since $\mathrm{im}(\phi)$ keeps changing during the procedure, we cannot have direct control over it.) Thus, breaking up the tree into tiny shrubs in the (τk)-fine partition was the key to successfully embedding it in this case.

4.6 Proof of Lemma 4.13

This subsection is devoted to the proof of Lemma 4.13. We give an overview of our decomposition procedure. We start by extracting the edges of as many $(\gamma k, k)$-dense spots from G as possible; these together with the incident vertices will form the auxiliary graph $G_{\mathcal{D}}$. Most of the remaining edges will form the edge set of the graph G_{exp}. Next, we consider the intersections of the dense spots captured in $G_{\mathcal{D}}$. To the subgraph of $G_{\mathcal{D}}$ that is spanned by the large intersections we apply the Regularity Lemma for locally dense graphs (Lemma 2.13), and thus obtain G_{reg}. The other part of $V(G_{\mathcal{D}})$ will be taken as the $(\Lambda, \varepsilon, \gamma, k)$-avoiding set $\mathfrak{A}$.

Setting up the parameters. We start by setting

$$\tilde{\nu} := \varepsilon \cdot 3^{-\frac{\Omega\Lambda}{\gamma^3}}.$$

Let q_{MAXCL} be given by Lemma 2.13 for input parameters

$$m_{\triangleright \mathrm{L}2.13} := \frac{\Omega}{\gamma\tilde{\nu}} \quad , \quad z_{\triangleright \mathrm{L}2.13} := 4s \quad \text{and} \quad \varepsilon_{\triangleright \mathrm{L}2.13} := \varepsilon\ . \tag{4.7}$$

Define an auxiliary parameter $q := \max\{q_{\mathrm{MAXCL}}, \varepsilon^{-1}\}$ and choose the output parameters of Lemma 4.13 as

$$k_0 := \left\lceil \frac{q_{\mathrm{MAXCL}}}{\tilde{\nu}} \right\rceil \quad \text{and} \quad \nu := \frac{\tilde{\nu}}{q}.$$

Defining $\mathcal{D}$ and G_{exp}. Given a graph G, take a set $\mathcal{D}$ of edge-disjoint $(\gamma k, \gamma)$-dense spots such that the resulting graph $G_{\mathcal{D}} \subseteq G$ (which contains those vertices and edges that are contained in $\bigcup \mathcal{D}$) has a maximal number of edges.

Then by Lemma 2.3 there exists a graph $G_{\mathrm{exp}} \subseteq G - G_{\mathcal{D}}$ with $\deg^{\min}(G_{\mathrm{exp}}) > \rho k$ and such that

$$|E(G) \setminus (E(G_{\mathrm{exp}}) \cup E(G_{\mathcal{D}}))| \leqslant \rho k n \ . \tag{4.8}$$

This choice of $\mathcal{D}$ and G_{exp} already satisfies Properties 4 and 7 of Definition 4.7.

Preparing for an application of the Regularity Lemma. Let

$$\mathcal{X} := \boxplus_D \{U, W, V(G) \setminus V(D)\} \ ,$$

where the partition refinement ranges over all $D = (U, W; F) \in \mathcal{D}$. Let $\mathcal{B} := \{X \in \mathcal{X} \ : \ X \subseteq V(G_{\mathcal{D}})\}$, $\tilde{\mathcal{B}} := \{B \in \mathcal{B} \ : \ |B| > 2\tilde{\nu}k\}$, and $\tilde{\mathcal{C}} := \mathcal{B} \setminus \tilde{\mathcal{B}}$. Furthermore let $\tilde{B} := \bigcup_{B \in \tilde{\mathcal{B}}} B$ and $\mathfrak{A} := \bigcup_{C \in \tilde{\mathcal{C}}} C$. Let $V_{\rightsquigarrow \mathfrak{A}} := \{v \in V(G) \ : \ \deg(v, \mathfrak{A}) > b\}$.

Now, partition each set $B \in \tilde{\mathcal{B}}$ into $c_B := \lceil |B| / 2\tilde{\nu}k \rceil$ sets $B_1, \ldots, B_{c_B}$ of cardinalities differing by at most one, and let $\mathcal{B}'$ be the set containing all the sets B_i (for all $B \in \tilde{\mathcal{B}}$). Then for each $B \in \mathcal{B}'$ we have that

$$\tilde{\nu}k \leqslant |B| \leqslant 2\tilde{\nu}k \leqslant \varepsilon k \ . \tag{4.9}$$

Construct a graph H on $\mathcal{B}'$ by making two vertices $A_1, A_2 \in \mathcal{B}'$ adjacent in H if

(A) there is a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $A_1 \subseteq U$ and $A_2 \subseteq W$, and

(B) $\mathrm{d}_G(A_1, A_2) \geqslant \gamma$.

Note that it follows from the way $\mathcal{D}$ was chosen that if $A_1 A_2 \in E(H)$ then $G[A_1, A_2] = G_{\mathcal{D}}[A_1, A_2]$. But on the other hand note that we do not necessarily have $G[A_1, A_2] = D[A_1, A_2]$ for the dense spot D appearing in (A); just because there may be several such dense spots D.

By assumption of Lemma 4.13, $\deg^{\max}(G) \leqslant \Omega k$. So, for each $B \in \mathcal{B}'$ we have $e_G(B, \tilde{B} \setminus B) \leqslant \Omega k |B|$. On the other hand, (4.9) and (B) imply that $\gamma \tilde{\nu} k |B| \deg_H(B) \leqslant e_G(B, \tilde{B} \setminus B)$. We conclude that

$$\deg^{\max}(H) \leqslant \frac{\Omega}{\gamma \tilde{\nu}} = m_{\triangleright \mathrm{L}2.13} \ . \tag{4.10}$$

Regularising the dense spots in $\tilde{B}$. We use Lemma 2.13 with parameters $m_{\triangleright \mathrm{L}2.13}, z_{\triangleright \mathrm{L}2.13}$ and $\varepsilon_{\triangleright \mathrm{L}2.13}$ as defined by (4.7) on the graphs $H_{\triangleright \mathrm{L}2.13} := G_{\mathcal{D}}$ and $F_{\triangleright \mathrm{L}2.13} := H$, together with the ensemble $\mathcal{B}'$ in the role of the sets W_i, and partition of $V(G_{\mathcal{D}})$ induced by

$$\mathcal{Z}_{\triangleright \mathrm{L}2.13} := \mathcal{V} \boxplus \left\{ V(G_{\mathrm{exp}}), V(G) \setminus V(G_{\mathrm{exp}}) \right\} \boxplus \left\{ V_{\rightsquigarrow \mathfrak{A}}, V(G) \setminus V_{\rightsquigarrow \mathfrak{A}} \right\} \ .$$

Observe that $\mathcal{B}'$ is an $(\tilde{\nu}k)$-ensemble satisfying condition (2.3) of Lemma 2.13, by (4.9), by the choice of k_0, and by (4.10). We thus obtain integers $\{p_A\}_{A \in \mathcal{B}'}$ and a family $\mathbf{V} = \{W_A^{(1)}, \ldots, W_A^{(p_A)}\}_{A \in \mathcal{B}'}$ and a set $W_0 := \bigcup_{A \in \mathcal{B}'} W_A^{(0)}$ such that in particular we have the following.

(I) We have $\varepsilon^{-1} \leqslant p_A \leqslant q_{\mathrm{MAXCL}}$ for all $A \in \mathcal{B}'$.

(II) We have $|W_A^{(a)}| = |W_B^{(b)}|$ for any $A, B \in \mathcal{B}'$ and for any $a \in [p_A]$, $b \in [p_B]$.

(III) For any $A \in \mathcal{B}'$ and any $a \in [p_A]$, there is $V \in \mathcal{V}$ such that $W_A^{(a)} \subseteq V$. We either have that $W_A^{(a)} \subseteq V(G_{\mathrm{exp}})$, or $W_A^{(a)} \cap V(G_{\mathrm{exp}}) = \emptyset$ and $W_A^{(a)} \subseteq V_{\leadsto\mathfrak{A}}$, or $W_A^{(a)} \cap V_{\leadsto\mathfrak{A}} = \emptyset$.

(IV) $\sum_{e \in E(H)} |\mathrm{irreg}(e)| \leqslant \varepsilon \sum_{AB \in E(H)} |A||B|$, where $\mathrm{irreg}(AB)$ is the set of all edges of the graph G contained in an ε-irregular pair $(W_A^{(a)}, W_B^{(b)})$, with $a \in [p_A]$, $b \in [p_B]$, $AB \in E(H)$.

Let G_{reg} be obtained from $G_{\mathcal{D}}$ by erasing all vertices in W_0, and all edges that lie in pairs $(W_A^{(a)}, W_B^{(b)})$ which are irregular or of density at most γ^2. Then Properties 1, 2, 5 and 6 of Definition 4.7 are satisfied. Further, Lemma 2.14 implies (4.4).

Note that Properties (I), (II) and (4.9) imply that for all $A \in \mathcal{B}'$ and for any $a \in [p_A]$ we have that

$$\varepsilon k \geqslant |A| \geqslant |W_A^{(a)}| \geqslant \frac{\tilde{\nu} k}{q_{\mathrm{MAXCL}}} \geqslant \frac{\tilde{\nu} k}{q} = \nu k.$$

Thus also Property 3 of Definition 4.7 holds.

Furthermore, by (4.8) and (4.4), the number of edges that are not captured by $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ is at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma + \rho)kn$.

So, it only remains to see Property 8 of Definition 4.7.

The avoiding property of $\mathfrak{A}$. In order to see Property 8 of Definition 4.7, we have to show that $\mathfrak{A}$ is $(\Lambda, \varepsilon, \gamma, k)$-avoiding with respect to $\mathcal{D}$. For this, let $\bar{U} \subseteq V(G)$ be such that $|\bar{U}| \leqslant \Lambda k$. Let X be the set of those vertices $v \in \mathfrak{A}$ that are not contained in any dense spot $D \in \mathcal{D}$ for which $|\bar{U} \cap V(D)| \leqslant \gamma^2 k$. Our aim is to see that $|X| \leqslant \varepsilon k$.

Let $\mathcal{D}_X \subseteq \mathcal{D}$ be the set of all dense spots D with $X \cap V(D) \neq \emptyset$. Setting $\mathcal{A} := \{A \in \tilde{\mathcal{C}} : A \cap X \neq \emptyset\}$, the definition of $\mathfrak{A}$ trivially implies that $\frac{|X|}{2\tilde{\nu}k} \leqslant |\mathcal{A}|$. Now, by the definition of $\mathcal{B}$, we know that there are at most $3^{|\mathcal{D}_X|}$ sets $A \in \mathcal{A}$. Indeed, for each $D = (U, W; F) \in \mathcal{D}_X$, either A is a subset of U, or of W, or of $V(G) \setminus V(D)$. Thus,

$$3^{|\mathcal{D}_X|} \geqslant |\mathcal{A}| \geqslant \frac{|X|}{\tilde{\nu} k} \; . \tag{4.11}$$

By Fact 4.4, each vertex of $V(G)$ lies in at most Ω/γ of the $(\gamma k, \gamma)$-dense spots from $\mathcal{D}$. Hence

$$\frac{\Omega}{\gamma}|\bar{U}| \geqslant \sum_{D \in \mathcal{D}_X} |V(D) \cap \bar{U}| \geqslant |\mathcal{D}_X| \gamma^2 k \overset{(4.11)}{\geqslant} \log_3 \left(\frac{|X|}{\tilde{\nu} k} \right) \gamma^2 k \; ,$$

where the second inequality holds by the definition of X. Thus

$$|X| \leqslant 3^{\frac{\Omega\Lambda}{\gamma^3}} \cdot \tilde{\nu} k = \varepsilon k \; ,$$

as desired. This finishes the proof of Lemma 4.13.

Remark 4.15. *The bounded decomposition given by Lemma 4.13 is not uniquely determined, and can actually vary vastly. This is caused by the arbitrariness in the choice of the dense spots from which we obtain the cluster graph G_{reg}.*

This situation is an acute contrast with the situation of decomposition of dense graphs (which is given by the Szemerédi Regularity Lemma). Indeed, in the dense setting the structure of the cluster graph is essentially unique, cf. [ASS09].[9]

Of course, the ambiguity of the bounded decomposition of G propagates to Lemma 4.14. We will have to deal with implications of this ambiguity in Section 6.

4.7 Lemma 4.13 algorithmically

Let us look back at the proof of Lemma 4.13 and see that we can get a bounded decomposition of any bounded-degree graph algorithmically in quasipolynomial time (in the order of the graph). Note that this in turn provides efficiently a sparse classification of any graph since the initial step of splitting the graph into huge degree vertices and bounded degree (cf. Lemma 4.1) can be done in polynomial time.

There are only two steps in the proof of Lemma 4.13 which need to be done algorithmically: the extraction of dense spots, and the simultaneous regularization of some dense pairs.

It will be more convenient to work with a relaxation of the notion of dense spots. We call a graph H *(d, ℓ)-thick* if $v(H) \geqslant \ell$, and $e(H) \geqslant dv(H)^2$. Thick graphs are a relaxation of dense spots, where the minimum degree condition is replaced by imposing a lower bound on the order, and the bipartiteness requirement is dropped. It can be verified that in our proof it is not important that the dense spots $\mathcal{D}$ and the nowhere-dense graph G_{exp} are parametrized by the same constants, i.e., the entire proof would go through even if the spots in $\mathcal{D}$ were $(\gamma k, \gamma)$-dense, and G_{exp} was $(\beta k, \beta)$-nowhere-dense for some $\beta \gg \gamma$. Each $(\beta k, \beta)$-thick graph gives (algorithmically) a $(\beta k/4, \beta/4)$-dense spot, and thus it is enough to extract thick graphs.

For the extraction of thick graphs we would need to efficiently answer the following: Given a number $\beta > 0$ find a number $\gamma > 0$ such that for an input number h and an N-vertex graph we can localize in G a (γ, h)-thick graph if it contains a (β, h)-thick graph, or output NO otherwise.[10] Employing techniques from a deep paper of Arora, Frieze and Kaplan [AFK02], one can solve this problem in quasipolynomial time $O(N^{c \cdot \log N})$. This was communicated to us by Maxim Sviridenko. On the negative side, a truly polynomial algorithm seems to be out of reach as Alon, Arora, Manokaran, Moshovitz, and Weinstein [AAM$^+$] reduced the problem to the notorious hidden clique problem whose tractability has been open for twenty years.

Theorem 4.16 (Alon et al. [AAM$^+$])**.** *If there is no polynomial time algorithm for solving the clique problem for a planted clique of size $n^{1/3}$ then for any $\varepsilon \in (0, 1)$ and $\delta > 0$ there is no*

[9] The setting needs to be somewhat strengthened as otherwise there are counterexamples to uniqueness; compare Theorem 1 and Theorem 2 in [ASS09]. However morally this is true because of the uniqueness of graph limits [BCL09].

[10] We could additionally assume that $\deg^{\max}(G) \leqslant O(h)$ due to the previous step of removing the set $\mathbf{\Psi}$ of huge degree vertices.

polynomial time algorithm that distinguishes between a graph G on N vertices containing a clique of size $\kappa = N^{\varepsilon}$ and a graph G' on N vertices in which the densest subgraph on κ vertices has density at most δ.[11]

Of course, Theorem 4.16 leaves some hope for a polynomial time algorithm when $h = N^{o(1)}$ (which corresponds to $k_{\triangleright\mathrm{L}4.13} = n_{\triangleright\mathrm{L}4.13}^{o(1)}$).

The regularity lemma can be made algorithmic [ADL+94]. The algorithm from [ADL+94] is based on index pumping-up, and thus applies even to the locally dense setting of Lemma 2.13.

It will turn out that the extraction of dense spots is the only obstruction to a polynomial time algorithm for Theorem 1.3. In Section 10.1 we sketch a truly polynomial time algorithm which avoids this step. It seems that the method sketched there is generally applicable for problems which employ sparse classifications.

5 Augmenting a matching

In previous papers [AKS95, Zha11, PS12, Coo09, HP] concerning the LKS Conjecture in the dense setting the crucial turn was to find a matching in the cluster graph of the host graph possessing certain properties. We will prove a similar "structural result" in Section 6. In the present section, we prove the main tool for Section 6, namely Lemma 5.10. All preceding statements are only preparatory. The only exception is (the easy) Lemma 5.6 which is recycled later, in Section 7.

5.1 Dense spots and semiregular matchings

We need two definitions concerning graphs covered by dense spots.

Definition 5.1 (**(m,γ)-dense cover**)**.** *A* (m,γ)-dense cover *of a graph G is a family $\mathcal{D}$ of edge-disjoint (m,γ)-dense spots such that $E(G) = \bigcup_{D\in\mathcal{D}} E(D)$.*

Definition 5.2 ($\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ and $\bar{\mathcal{G}}(n,k,\Omega,\rho,\nu)$)**.** *We define $\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ to be the class of all tuples $(G,\mathcal{D},H,\mathcal{A})$ with the following properties:*

(i) G is a graph of order n with $\deg^{\max}(G) \leqslant \Omega k$,

(ii) H is a bipartite subgraph of G with colour classes A_H and B_H and with $e(H) \geqslant \tau k n$,

(iii) $\mathcal{D}$ is a $(\rho k,\rho)$-dense cover of G,

(iv) $\mathcal{A}$ is a (νk)-ensemble in G, and $A_H \subseteq \bigcup \mathcal{A}$,

(v) $A \cap U \in \{\emptyset, A\}$ for each $A \in \mathcal{A}$ and for each $D = (U,W;F) \in \mathcal{D}$.

[11]The result as stated in [AAM+] covers only the range $\varepsilon \in (\frac{1}{3}, 1)$. However there is a simple reduction by taking many disjoint copies of the general range to the restricted one.

Those G, $\mathcal{D}$ and $\mathcal{A}$ for which all conditions but (ii) *and the last part of* (iv) *hold will make up the triples $(G, \mathcal{D}, \mathcal{A})$ of the class $\bar{\mathcal{G}}(n, k, \Omega, \rho, \nu)$.*

We now prove our first auxiliary lemma on our way towards Lemma 5.10.

Lemma 5.3. *For every $\Omega \in \mathbb{N}$ and $\varepsilon, \rho, \tau > 0$ there is a number $\alpha > 0$ such that for every $\nu \in (0, 1)$ there exists a number $k_0 \in \mathbb{N}$ such that for each $k > k_0$ the following holds.*

For every $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n, k, \Omega, \rho, \nu, \tau)$ there are $(U, W; F) \in \mathcal{D}$, $A \in \mathcal{A}$ and $X, Y \subseteq V(G)$ such that

1) $|X| = |Y| > \alpha\nu k$,

2) $X \subseteq A \cap U \cap A_H$ and $Y \subseteq W \cap B_H$, where A_H and B_H are the colour classes of H, and

3) (X, Y) is an ε-regular pair in G of density $\mathrm{d}(X, Y) \geqslant \frac{\tau\rho}{4\Omega}$.

Proof. Let Ω, ε, ρ and τ be given. Applying Lemma 2.12 to $\varepsilon_{\triangleright \mathrm{L}2.12} := \min\{\varepsilon, \frac{\rho^2}{8\Omega}\}$ and $\ell_{\triangleright \mathrm{L}2.12} := 2$, we obtain numbers n_0 and M. We set

$$\alpha := \frac{\tau\rho}{\Omega^2 M}, \tag{5.1}$$

and given $\nu \in (0, 1)$, we set

$$k_0 := \frac{2n_0}{\alpha\nu M}.$$

Now suppose we are given $k > k_0$ and $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n, k, \Omega, \rho, \nu, \tau)$.

Property (i) of Definition 5.2 gives that $e(G) \leqslant \Omega kn/2$, and Property (ii) says that $e(H) \geqslant \tau kn$. So $e(H)/e(G) \geqslant 2\tau/\Omega$. Averaging, we find a dense spot $D = (U, W; F) \in \mathcal{D}$ such that

$$e_D(A_H, B_H) = |F \cap E(H)| \geqslant \frac{e(H)}{e(G)}|F| \geqslant \frac{2\tau|F|}{\Omega} \ . \tag{5.2}$$

Without loss of generality, we assume that

$$e_D(U \cap A_H, W \cap B_H) \geqslant \frac{1}{2} \cdot e_D(A_H, B_H) \geqslant e_D(U \cap B_H, W \cap A_H) \ , \tag{5.3}$$

as otherwise one can just interchange the roles of U and W. Then,

$$e_G(U \cap A_H, W \cap B_H) \overset{(5.3)}{\geqslant} \frac{1}{2} \cdot e_D(A_H, B_H) \overset{(5.2)}{\geqslant} \frac{\tau}{\Omega} \cdot |F|. \tag{5.4}$$

Let $\mathcal{A}' \subseteq \mathcal{A}$ denote the set of those $A \in \mathcal{A}$ with $0 < e_G(A \cap U \cap A_H, W \cap B_H) < \frac{\tau}{\Omega} \cdot |F| \cdot \frac{|A|}{|U|}$. Note that for each $A \in \mathcal{A}'$ we have $A \subseteq U$ by Definition 5.2 (v). Therefore,

$$e_G\left(\bigcup \mathcal{A}' \cap U \cap A_H, W \cap B_H\right) < \ \frac{\tau}{\Omega} \cdot |F| \cdot \frac{|\mathcal{A}'|}{|U|} \leqslant \ \frac{\tau}{\Omega} \cdot |F| \overset{(5.4)}{\leqslant} \ e_G(U \cap A_H, W \cap B_H) \ .$$

As $\mathcal{A}$ covers A_H, G has an edge xy with $x \in U \cap A_H \cap A$ for some $A \in \mathcal{A} \setminus \mathcal{A}'$ and $y \in W \cap B_H$. Set $X' := A \cap U \cap A_H = A \cap A_H$ and $Y' := W \cap B_H$. Then directly from the definition of $\mathcal{A}'$ and since D is a $(\rho k, \rho)$-dense spot, we obtain that

$$\mathrm{d}_G(X', Y') = \frac{e_G(X', Y')}{|X'||Y'|} \geqslant \frac{\frac{\tau}{\Omega} \cdot |F| \cdot \frac{|A|}{|U|}}{|A||W|} > \ \frac{\tau\rho}{\Omega}. \tag{5.5}$$

Also, since $(U, W; F) \in \mathcal{D}$, we have

$$|F| \geqslant \rho k |U| \ . \tag{5.6}$$

This enables us to bound the size of X' as follows.

$$\begin{aligned} |X'| &\geqslant \frac{e_G(X', Y')}{\deg^{\max}(G)} \\ \text{(as } A \notin \mathcal{A}' \text{ and by D5.2(i))} \quad &\geqslant \frac{\frac{\tau}{\Omega} \cdot \frac{|F|}{|U|} \cdot |A|}{\Omega k} \\ \text{(by (5.6))} \quad &\geqslant \frac{\tau \cdot \rho k \cdot |A|}{\Omega^2 k} \\ &\geqslant \frac{\tau \rho \nu k}{\Omega^2} \\ &\overset{(5.1)}{=} \alpha \nu k M \ . \end{aligned} \tag{5.7}$$

In the same way we see that

$$|Y'| \geqslant \ \alpha \nu k M \ . \tag{5.8}$$

Applying Lemma 2.12 to $G[X', Y']$ with prepartition $\{X', Y'\}$ we obtain a collection of sets $\mathcal{C} = \{C_i\}_{i=0}^p$, with $p < M$. By (5.7), and (5.8), we have that $|C_i| \geqslant \alpha\nu k$ for every $i \in [p]$. It is easy to deduce from (5.5) that there is at least one $\varepsilon_{\triangleright\mathrm{L}2.12}$-regular (and thus ε-regular) pair (X, Y), $X, Y \in \mathcal{C} \setminus \{C_0\}$, $X \subseteq X'$, $Y \subseteq Y'$ with $\mathrm{d}(X, Y) \geqslant \frac{\tau\rho}{4\Omega}$. Indeed, it suffices to count the number of edges incident with C_0, lying in $\varepsilon_{\mathrm{L}2.12}$-irregular pairs or belonging to too sparse pairs. These are strictly less than

$$(\varepsilon_{\triangleright\mathrm{L}2.12} + \varepsilon_{\triangleright\mathrm{L}2.12} + \frac{\rho^2}{4\Omega})|X||Y| \leqslant \frac{\rho^2}{2\Omega}|X||Y| \overset{(5.5)}{\leqslant} e(X', Y')$$

many, and thus not all edges between X' and Y'. This finishes the proof of Lemma 5.3. □

Instead of just one pair (X, Y), as it is given by Lemma 5.3, we shall later need several disjoint pairs. This motivates the following definition.

Definition 5.4 (**(ε, d, ℓ)-semiregular matching).** *A collection $\mathcal{N}$ of pairs (A, B) with $A, B \subseteq V(H)$ is called an* (ε, d, ℓ)-semiregular matching *of a graph H if*

(i) $|A| = |B| \geqslant \ell$ for each $(A, B) \in \mathcal{N}$,

(ii) (A, B) induces in H an ε-regular pair of density at least d, for each $(A, B) \in \mathcal{N}$, and

(iii) all involved sets A and B are pairwise disjoint.

Sometimes, when the parameters do not matter (as for instance in Definition 5.7 below) we write lazily semiregular matching.

For a semiregular matching $\mathcal{N}$, we shall write $\mathcal{V}_1(\mathcal{N}) := \{A : (A,B) \in \mathcal{N}\}$, $\mathcal{V}_2(\mathcal{N}) := \{B : (A,B) \in \mathcal{N}\}$ and $\mathcal{V}(\mathcal{N}) := \mathcal{V}_1(\mathcal{N}) \cup \mathcal{V}_2(\mathcal{N})$. Furthermore, we set $V_1(\mathcal{N}) := \bigcup \mathcal{V}_1(\mathcal{N})$, $V_2(\mathcal{N}) := \bigcup \mathcal{V}_2(\mathcal{N})$ and $V(\mathcal{N}) := V_1(\mathcal{N}) \cup V_2(\mathcal{N}) = \bigcup \mathcal{V}(\mathcal{N})$. As these definitions suggest, the orientations of the pairs $(A,B) \in \mathcal{N}$ are important. The sets A and B are called $\mathcal{N}$*-vertices* and the pair (A,B) is a $\mathcal{N}$*-edge*.

We say that a semiregular matching $\mathcal{N}$ *absorbes* a semiregular matching $\mathcal{M}$ if for every $(S,T) \in \mathcal{M}$ there exists $(X,Y) \in \mathcal{N}$ such that $S \subseteq X$ and $T \subseteq Y$. In the same way, we say that a family of dense spots $\mathcal{D}$ *absorbes* a semiregular matching $\mathcal{M}$ if for every $(S,T) \in \mathcal{M}$ there exists $(U,W;F) \in \mathcal{D}$ such that $S \subseteq U$ and $T \subseteq W$.

We later need the following easy bound on the size of the elements of $\mathcal{V}(\mathcal{M})$.

Fact 5.5. *Suppose that $\mathcal{M}$ is an (ε, d, ℓ)-semiregular matching in a graph H. Then $|C| \leqslant \frac{\deg^{\max}(H)}{d}$ for each $C \in \mathcal{V}(\mathcal{M})$.*

Proof. Let for example $(C,D) \in \mathcal{M}$. The maximum degree of H is at least as large as the average degree of the vertices in D, which is at least $d|C|$. □

The next lemma, Lemma 5.6, is a second step towards Lemma 5.10. Whereas Lemma 5.3 gives one dense regular pair, in the same setting Lemma 5.6 provides us with a dense semiregular matching.

Lemma 5.6. *For every $\Omega \in \mathbb{N}$ and $\rho, \varepsilon, \tau \in (0,1)$ there exists $\alpha > 0$ such that for every $\nu \in (0,1)$ there is a number $k_0 \in \mathbb{N}$ such that the following holds for every $k > k_0$.*

For each $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ there exists an $(\varepsilon, \frac{\tau\rho}{8\Omega}, \alpha\nu k)$-semiregular matching $\mathcal{M}$ of G such that

(1) for each $(X,Y) \in \mathcal{M}$ there are $A \in \mathcal{A}$, and $D = (U,W;F) \in \mathcal{D}$ such that $X \subseteq U \cap A \cap A_H$ and $Y \subseteq W \cap B_H$, and

(2) $|V(\mathcal{M})| \geqslant \frac{\tau}{2\Omega} n$.

Proof. Let $\alpha := \alpha_{\triangleright \mathrm{L}5.3} > 0$ be given by Lemma 5.3 for the input parameters $\Omega_{\triangleright \mathrm{L}5.3} := \Omega$, $\varepsilon_{\triangleright \mathrm{L}5.3} := \varepsilon$, $\tau_{\triangleright \mathrm{L}5.3} := \tau/2$ and $\rho_{\triangleright \mathrm{L}5.3} := \rho$. Now, for $\nu_{\triangleright \mathrm{L}5.3} := \nu$, Lemma 5.3 yields a number $k_0 \in \mathbb{N}$.

Now let $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$. Let $\mathcal{M}$ be an inclusion-maximal $(\varepsilon\rho, \frac{\tau\rho}{8\Omega}, \alpha\nu k)$-semiregular matching with property (1). We claim that

$$e_G(A_H \setminus V_1(\mathcal{M}), B_H \setminus V_2(\mathcal{M})) < \frac{\tau}{2} kn. \tag{5.9}$$

Indeed, suppose otherwise. Then the bipartite subgraph H' of G induced by the sets $A_H \setminus V_1(\mathcal{M}) = A_H \setminus V(\mathcal{M})$ and $B_H \setminus V_2(\mathcal{M}) = B_H \setminus V(\mathcal{M})$ satisfies Property (ii) of Definition 5.2, with $\tau_{\triangleright \mathrm{D}5.2} := \tau/2$. So, we have that $(G, \mathcal{D}, H', \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau/2)$.

Thus Lemma 5.3 for $(G, \mathcal{D}, H', \mathcal{A})$ yields a dense spot $D = (U,W;F) \in \mathcal{D}$ and a set $A \in \mathcal{A}$, together with two sets $X \subseteq U \cap A \cap (A_H \setminus V(\mathcal{M}))$, $Y \subseteq W \cap (B_H \setminus V(\mathcal{M}))$ such that $|X| = |Y| >$

$\alpha_{\triangleright \mathrm{L}5.3}\nu k = \alpha\nu k$, and such that (X,Y) is $\varepsilon_{\triangleright \mathrm{L}5.3}$-regular and has density at least

$$\frac{\tau_{\triangleright \mathrm{L}5.3}\rho_{\triangleright \mathrm{L}5.3}}{4\Omega_{\triangleright \mathrm{L}5.3}} = \frac{\tau\rho}{8\Omega}.$$

As this contradicts the maximality of $\mathcal{M}$, we have shown (5.9).

In order to see (2), it suffices to observe that by (5.9) and by Property (ii) of Definition 5.2, the set $V(\mathcal{M})$ is incident with at least $\tau kn - \frac{\tau}{2}kn = \frac{\tau}{2}kn$ edges. By Definition 5.2 (i), it follows that $|V(\mathcal{M})| \geqslant \frac{\tau}{2}kn \cdot \frac{1}{\Omega k} \geqslant \frac{\tau}{2\Omega}n$, as desired. □

5.2 Augmenting paths for matchings

We now prove the main lemma of Section 5, namely Lemma 5.10. We will use an augmenting path technique for our semiregular matchings, similar to the augmenting paths commonly used for traditional matching theorems. For this, we need the following definitions.

Definition 5.7 (**Alternating path, augmenting path).** *Given an n-vertex graph G, and a semiregular matching $\mathcal{M}$, we call a sequence $\mathfrak{S} = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \dots, \mathcal{A}_h, Y_h)$ $(h \geqslant 0)$ an (δ, s)-*alternating path for $\mathcal{M}$ from Y_0 *if for all $i \in [h]$ we have*

(i) $\mathcal{A}_i \subseteq \mathcal{V}_1(\mathcal{M})$ and the sets $\mathcal{A}_i$ are pairwise disjoint,

(ii) $Y_0 \subseteq V(G) \setminus V(\mathcal{M})$ and $Y_i = \bigcup_{(A,B)\in\mathcal{M}, A\in\mathcal{A}_i} B$,

(iii) $|Y_{i-1}| \geqslant \delta n$, and

(iv) $e(A, Y_{i-1}) \geqslant s \cdot |A|$, for each $A \in \mathcal{A}_i$.

If in addition there is a set $\mathcal{C}$ of disjoint subsets of $V(G) \setminus (Y_0 \cup V(\mathcal{M}))$ such that

(v) $e(\bigcup \mathcal{C}, Y_h) \geqslant t \cdot n$,

*then we say that $\mathfrak{S}' = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \dots, \mathcal{A}_h, Y_h, \mathcal{C})$ is an (δ, s, t)-*augmenting path for $\mathcal{M}$ from Y_0 to $\mathcal{C}$.

The number h is called the the length *of $\mathfrak{S}$ (or of $\mathfrak{S}'$).*

Next, we show that a semiregular matching either has an augmenting path or admits a partition into two parts so that there are only few edges which cross these parts in a certain way.

Lemma 5.8. *Given an n-vertex graph G with $\deg^{\max}(G) \leqslant \Omega k$, a number $\tau \in (0,1)$, a semiregular matching $\mathcal{M}$, a set $Y_0 \subseteq V(G) \setminus V(\mathcal{M})$, and a set $\mathcal{C}$ of disjoint subsets of $V(G) \setminus (V(\mathcal{M}) \cup Y_0)$, one of the following holds:*

(M1) *There is a semiregular matching $\mathcal{M}'' \subseteq \mathcal{M}$ with $e\left(\bigcup \mathcal{C} \cup V_1(\mathcal{M} \setminus \mathcal{M}''), Y_0 \cup V_2(\mathcal{M}'')\right) < \tau nk$,*

(M2) *$\mathcal{M}$ has an $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$-augmenting path of length at most $2\Omega/\tau$ from Y_0 to $\mathcal{C}$.*

Proof. If $|Y_0| \leqslant \frac{\tau}{2\Omega}n$ then **(M1)** is satisfied for $\mathcal{M}'' := \emptyset$. Let us therefore assume otherwise.

Choose a $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k)$-alternating path $\mathfrak{S} = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_h, Y_h)$ for $\mathcal{M}$ with $|\bigcup_{\ell=1}^{h} \mathcal{A}_\ell|$ maximal.

Now, let $\ell^* \in \{0, 1, \ldots, h\}$ be maximal with $|Y_{\ell^*}| \geqslant \frac{\tau}{2\Omega}n$. Then $\ell^* \in \{h, h-1\}$. Moreover, as $|Y_\ell| \geqslant \frac{\tau}{2\Omega}n$ for all $\ell \leqslant \ell^*$, we have that $(\ell^*+1) \cdot \frac{\tau}{2\Omega}n \leqslant |\bigcup_{\ell \leqslant \ell^*} Y_\ell| \leqslant n$ and thus

$$\ell^* + 1 \leqslant \frac{2\Omega}{\tau}. \tag{5.10}$$

Let $\mathcal{M}'' \subseteq \mathcal{M}$ consist of all $\mathcal{M}$-edges $(A, B) \in \mathcal{M}$ with $A \in \bigcup_{\ell \in [h]} \mathcal{A}_\ell$. Then, by the choice of $\mathfrak{S}$,

$$\begin{aligned} e\left(V_1(\mathcal{M} \setminus \mathcal{M}''), \bigcup_{\ell=0}^{\ell^*} Y_\ell\right) &= \sum_{\ell=0}^{\ell^*} e\left(V_1(\mathcal{M} \setminus \mathcal{M}''), Y_\ell\right) \\ &< (\ell^*+1) \cdot \frac{\tau^2}{8\Omega}k \cdot |V_1(\mathcal{M} \setminus \mathcal{M}'')| \overset{(5.10)}{\leqslant} \frac{\tau}{4}kn. \end{aligned} \tag{5.11}$$

Furthermore, if $\ell^* = h - 1$ (that is, if $|Y_h| < \frac{\tau}{2\Omega}n$) then

$$e\left(V_1(\mathcal{M} \setminus \mathcal{M}'') \cup \bigcup \mathcal{C}, Y_h\right) \;<\; \frac{\tau}{2\Omega}n \cdot \mathrm{deg}^{\max}(G) \;\leqslant\; \frac{\tau}{2\Omega}\Omega kn \;=\; \frac{\tau}{2}kn. \tag{5.12}$$

So, regardless whether $h = \ell^*$ or $h = \ell^* + 1$, we get from (5.11) and (5.12) that

$$e\left(V_1(\mathcal{M} \setminus \mathcal{M}'') \cup \bigcup \mathcal{C}, Y_0 \cup V_2(\mathcal{M}'')\right) < \frac{3}{4}\tau kn + e\left(\bigcup \mathcal{C}, \bigcup_{\ell=0}^{\ell^*} Y_\ell\right).$$

Thus, if $e(\bigcup \mathcal{C}, \bigcup_{\ell=0}^{\ell^*} Y_\ell) \leqslant \frac{\tau}{4}kn$, we see that (**M1**) satisfied for $\mathcal{M}''$. So, assume otherwise. Then, by (5.10), there is an index $j \in \{0, 1, \ldots, \ell^*\}$ so that

$$e\left(\bigcup \mathcal{C}, Y_j\right) \;>\; \frac{\tau^2}{16\Omega}kn,$$

and thus, $(Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_h, Y_h, \mathcal{C})$ is an $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$-augmenting path for $\mathcal{M}$. This shows **(M2)**. $\square$

Building on Lemma 5.6 and Lemma 5.8 we prove the following.

Lemma 5.9. *For every $\Omega \in \mathbb{N}$ and $\tau \in (0, \frac{1}{2\Omega})$ there is a number $\tau' \in (0, \tau)$ such that for every $\rho \in (0,1)$ there is a number $\alpha \in (0, \tau'/2)$ such that for every $\varepsilon \in (0, \alpha)$ there is a number $\pi > 0$ such that for every $\gamma > 0$ there is $k_0 \in \mathbb{N}$ such that the following holds for every $k > k_0$ and every $h \in (\gamma k, k/2)$.*

Let G be a graph of order n with $\mathrm{deg}^{\max}(G) \leqslant \Omega k$, with an (ε^3, ρ, h)-semiregular matching $\mathcal{M}$ and with a $(\rho k, \rho)$-dense cover $\mathcal{D}$ that absorbs $\mathcal{M}$. Let $Y \subseteq V(G) \setminus V(\mathcal{M})$, and let $\mathcal{C}$ be an h-ensemble in G outside $V(\mathcal{M}) \cup Y$. Assume that $U \cap C \in \{\emptyset, C\}$ for each $D = (U, W; F) \in \mathcal{D}$ and each $C \in \mathcal{C} \cup \mathcal{V}_1(\mathcal{M})$.

Then one of the following holds.

(I) There is a semiregular matching $\mathcal{M}'' \subseteq \mathcal{M}$ such that

$$e\left(\bigcup \mathcal{C} \cup V_1(\mathcal{M} \setminus \mathcal{M}''), Y \cup V_2(\mathcal{M}'')\right) < \tau nk.$$

(II) There is an $(\varepsilon, \alpha, \pi h)$-semiregular matching $\mathcal{M}'$ such that

(C1) *$|V(\mathcal{M}) \setminus V(\mathcal{M}')| \leqslant \varepsilon n$, and $|V(\mathcal{M}')| \geqslant |V(\mathcal{M})| + \frac{\tau'}{2}n$, and*

(C2) *for each $(T, Q) \in \mathcal{M}'$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$.*

Proof. We divide the proof into five steps.

Step 1: Setting up the parameters. Suppose that Ω and τ are given. For $\ell = 0, 1, \ldots, \lceil 2\Omega/\tau \rceil$, we define the auxiliary parameters

$$\tau^{(\ell)} := \left(\frac{\tau^2}{32\Omega}\right)^{\lceil \frac{2\Omega}{\tau} \rceil - \ell + 2}, \tag{5.13}$$

and set

$$\tau' := \frac{\tau^{(0)}}{2\Omega}.$$

Given ρ, we define

$$\alpha := \frac{\tau' \rho}{16\Omega}.$$

Then, given ε, for $\ell = 0, 1, \ldots, \lceil 2\Omega/\tau \rceil$, we define the further auxiliary parameters

$$\mu^{(\ell)} := \alpha_{\triangleright \mathrm{L}5.6}\big(\Omega, \rho, \varepsilon^3, \tau^{(\ell)}\big)$$

which are given by Lemma 5.6 for input parameters $\Omega_{\triangleright \mathrm{L}5.6} := \Omega$, $\rho_{\triangleright \mathrm{L}5.6} := \rho$, $\varepsilon_{\triangleright \mathrm{L}5.6} := \varepsilon^3$, and $\tau_{\triangleright \mathrm{L}5.6} := \tau^{(\ell)}$. Set

$$\pi := \frac{\varepsilon}{2} \cdot \min\left\{\mu^{(\ell)} \,:\, \ell = 0, \ldots, \lceil 2\Omega/\tau \rceil\right\},$$

Given the next input parameter γ, Lemma 5.6 for parameters as above and the final input $\nu_{\triangleright \mathrm{L}5.6} := \gamma$ yields $k_{0_{\triangleright \mathrm{L}5.6}} =: k_0^{(\ell)}$. Set

$$k_0 := \max\left\{k_0^{(\ell)} \,:\, \ell = 0, \ldots, \lceil 2\Omega/\tau \rceil\right\}.$$

Step 2: Finding an augmenting path. We apply Lemma 5.8 to G, τ, $\mathcal{M}$, Y and $\mathcal{C}$. Since (**M1**) corresponds to (I), let us assume that the outcome of the lemma is (**M2**). Then there is a $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$-augmenting path $\mathfrak{S}' = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_{j^*}, Y_{j^*}, \mathcal{C})$ for $\mathcal{M}$ starting from $Y_0 := Y$ such that $j^* \leqslant 2\Omega/\tau$.

Our aim is now to show that (II) holds.

Step 3: Creating parallel matchings. Inductively, for $\ell = j^*, j^* - 1, \ldots, 0$ we shall define auxiliary bipartite induced subgraphs $H^{(\ell)} \subseteq G$ with colour classes $P^{(\ell)}$ and Y_ℓ that satisfy

(a) $e(H^{(\ell)}) \geqslant \tau^{(\ell)} kn$,

and $(\varepsilon^3, 2\alpha, \mu^{(\ell)} h)$-semiregular matchings $\mathcal{M}^{(\ell)}$ that satisfy

(b) $V_1(\mathcal{M}^{(\ell)}) \subseteq P^{(\ell)}$,

(c) for each $(A', B') \in \mathcal{M}^{(\ell)}$ there are a dense spot $(U, W; F) \in \mathcal{D}$ and a set $A \in \mathcal{V}_1(\mathcal{M})$ (or a set $A \in \mathcal{C}$ if $\ell = j^*$) such that $A' \subseteq U \cap A$ and $B' \subseteq W \cap Y_\ell$,

(d) $|V(\mathcal{M}^{(\ell)})| \geqslant \frac{\tau^{(\ell)}}{2\Omega} n$, and

(e) $|B \cap V_2(\mathcal{M}^{(\ell)})| = |A \cap P^{(\ell-1)}|$ for each edge $(A, B) \in \mathcal{M}$, if $\ell > 0$.

We take $H^{(j^*)}$ as the induced bipartite subgraph of G with colour classes $P^{(j^*)} := \bigcup \mathcal{C}$ and Y_{j^*}. Definition 5.7 (v) together with (5.13) ensures (a) for $\ell = j^*$. Now, for $\ell \leqslant j^*$, suppose $H^{(\ell)}$ is defined already. Further, if $\ell < j^*$ suppose also that $\mathcal{M}^{(\ell+1)}$ is defined already. We shall define $\mathcal{M}^{(\ell)}$, and, if $\ell > 0$, we shall also define $H^{(\ell-1)}$.

Observe that $(G, \mathcal{D}, H^{(\ell)}, \mathcal{A}_\ell) \in \mathcal{G}(n, k, \Omega, \rho, \frac{h}{k}, \tau^{(\ell)})$, because of (a) and the assumptions of the lemma. So, applying Lemma 5.6 to $(G, \mathcal{D}, H^{(\ell)}, \mathcal{A}_\ell)$ and noting that $\frac{\tau^{(\ell)}\rho}{8\Omega} \geqslant 2\alpha$ we obtain an $(\varepsilon^3, 2\alpha, \mu^{(\ell)} h)$-semiregular matching $\mathcal{M}^{(\ell)}$ that satisfies conditions (b)–(d).

If $\ell > 0$, we define $H^{(\ell-1)}$ as follows. For each $(A, B) \in \mathcal{M}$ take a set $\tilde{A} \subseteq A$ of cardinality $|\tilde{A}| = |B \cap V(\mathcal{M}^{(\ell)})|$ so that

$$e(\tilde{A}, Y_{\ell-1}) \ \geqslant \ \frac{\tau^2}{8\Omega} k \cdot |\tilde{A}| \ . \tag{5.14}$$

This is possible by Definition 5.7 (iv): just choose those vertices from A for $\tilde{A}$ that send most edges to $Y_{\ell-1}$. Let $P^{(\ell-1)}$ be the union of all the sets $\tilde{A}$. Then, (e) is satisfied. Furthermore,

$$|P^{(\ell-1)}| \ = \ |V_2(\mathcal{M}^{(\ell)})| \ \overset{\text{(d)}}{\geqslant} \ \frac{\tau^{(\ell)}}{4\Omega} n.$$

So, by (5.14),

$$e(P^{(\ell-1)}, Y_{\ell-1}) \ \geqslant \ \frac{\tau^2}{8\Omega} k \cdot |P^{(\ell-1)}| \ \geqslant \ \frac{\tau^2 \cdot \tau^{(\ell)}}{32\Omega^2} kn \ \overset{(5.13)}{=} \ \tau^{(\ell-1)} kn \ . \tag{5.15}$$

We let $H^{(\ell-1)}$ be the bipartite subgraph of G induced by the colour classes $P^{(\ell-1)}$ and $Y_{\ell-1}$. Then (5.15) establishes (a) for $H^{(\ell-1)}$. This finishes step ℓ.[12]

[12] Recall that the matching $\mathcal{M}^{(\ell-1)}$ is only to be defined in step $\ell - 1$.

Step 4: Harmonising the matchings. Our semiregular matchings $\mathcal{M}^{(0)}, \ldots, \mathcal{M}^{(j^*)}$ will be a good base for constructing the semiregular matching $\mathcal{M}'$ we are after. However, we do not know anything about $|B \cap V_2(\mathcal{M}^{(\ell)})| - |A \cap V_1(\mathcal{M}^{(\ell-1)})|$ for the $\mathcal{M}$-edges $(A, B) \in \mathcal{M}$. But this term will be crucial in determining how much of $V(\mathcal{M})$ gets lost when we replace some of its $\mathcal{M}$-edges with $\bigcup \mathcal{M}^{(\ell)}$-edges. For this reason, we refine $\mathcal{M}^{(\ell)}$ in a way that its $\mathcal{M}^{(\ell)}$-edges become almost equal-sized.

Formally, we shall inductively construct semiregular matchings $\mathcal{N}^{(0)}, \ldots, \mathcal{N}^{(j^*)}$ such that for $\ell = 0, \ldots, j^*$ we have

(A) $\mathcal{N}^{(\ell)}$ is an $(\varepsilon, \alpha, \pi h)$-semiregular matching,

(B) $\mathcal{M}^{(\ell)}$ absorbes $\mathcal{N}^{(\ell)}$,

(C) if $\ell > 0$ and $(A, B) \in \mathcal{M}$ with $A \in \mathcal{A}_\ell$ then $|A \cap V(\mathcal{N}^{(\ell-1)})| \geqslant |B \cap V(\mathcal{N}^{(\ell)})|$, and

(D) $|V_2(\mathcal{N}^{(\ell)})| \geqslant |V_1(\mathcal{N}^{(\ell-1)})| - \frac{\varepsilon}{2} \cdot |V_2(\mathcal{M}^{(\ell)})|$ if $\ell > 0$ and $|V_2(\mathcal{N}^{(0)})| \geqslant \frac{\tau^{(0)}}{2\Omega} n = \tau' n$.

Set $\mathcal{N}^{(0)} := \mathcal{M}^{(0)}$. Clearly (B) holds for $\ell = 0$, (A) is easy to check, and (C) is void. Finally, Property (D) holds because of (d). Suppose now $\ell > 0$ and that we already constructed matchings $\mathcal{N}^{(0)}, \ldots, \mathcal{N}^{(\ell-1)}$ satisfying Conditions (A)–(D).

Observe that for any $(A, B) \in \mathcal{M}$ we have that

$$|B \cap V_2(\mathcal{M}^{(\ell)})| \overset{\text{(b),(e)}}{\geqslant} |A \cap V_1(\mathcal{M}^{(\ell-1)})| \geqslant |A \cap V_1(\mathcal{N}^{(\ell-1)})|, \tag{5.16}$$

where the last inequality holds because of (B) for $\ell - 1$.

So, we can choose a subset $X^{(\ell)} \subseteq V_2(\mathcal{M}^{(\ell)})$ such that $|B \cap X^{(\ell)}| = |A \cap V(\mathcal{N}^{(\ell-1)})|$ for each $(A, B) \in \mathcal{M}$. Now, for each $(S, T) \in \mathcal{M}^{(\ell)}$ write $\widehat{T} := T \cap X^{(\ell)}$, and choose a subset $\widehat{S}$ of S of size $|\widehat{T}|$. Set

$$\mathcal{N}^{(\ell)} := \left\{ (\widehat{S}, \widehat{T}) \,:\, (S, T) \in \mathcal{M}^{(\ell)}, |\widehat{T}| \geqslant \frac{\varepsilon}{2} \cdot |T| \right\}.$$

Then (B) and (C) hold for ℓ.

For (A), note that Fact 2.7 implies that $\mathcal{N}^{(\ell)}$ is an $\left(\varepsilon, 2\alpha - \varepsilon^3, \frac{\varepsilon}{2}\mu^{(\ell)} h\right)$-semiregular matching.

In order to see (D), it suffices to observe that

$$\begin{aligned}
|V_2(\mathcal{N}^{(\ell)})| &= \sum_{(\widehat{S}, \widehat{T}) \in \mathcal{N}^{(\ell)}} |\widehat{T}| \\
&\geqslant |X^{(\ell)}| - \sum_{(S,T) \in \mathcal{M}^{(\ell)}} \frac{\varepsilon}{2} \cdot |T| \\
&\geqslant \sum_{(A,B) \in \mathcal{M}} |A \cap V_1(\mathcal{N}^{(\ell-1)})| - \frac{\varepsilon}{2} \cdot |V_2(\mathcal{M}^{(\ell)})| \\
&= |V_1(\mathcal{N}^{(\ell-1)})| - \frac{\varepsilon}{2} \cdot |V_2(\mathcal{M}^{(\ell)})|.
\end{aligned}$$

Step 5: The final matching. For each $\ell = 1, 2, \ldots, j^*$ let $\mathcal{L}$ denote the set of all $\mathcal{M}$-edges $(A, B) \in \mathcal{M}$ with $|A'| > \frac{\varepsilon}{2} \cdot |A|$, where $A' := A \setminus V_1(\mathcal{N}^{(\ell-1)})$. Further, for each $(A, B) \in \mathcal{M}$, choose a set $B' \subseteq B \setminus V_2(\mathcal{N}^{(\ell)})$ of cardinality $|A'|$. This is possible by (C). Set

$$\mathcal{K} := \{(A', B') : (A, B) \in \mathcal{L}\}.$$

By the assumption of the lemma, for every $(A', B') \in \mathcal{K}$ there are an edge $(A, B) \in \mathcal{M}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that

$$A' \subseteq A \subseteq U \text{ and } B' \subseteq B \subseteq W. \tag{5.17}$$

Since $\mathcal{M}$ is (ε^3, ρ, h)-semiregular we have by Fact 2.7 that $\mathcal{K}$ is a $(\varepsilon, \rho - \varepsilon^3, \frac{\varepsilon}{2}h)$-semiregular matching. Set

$$\mathcal{M}' := \mathcal{K} \cup \bigcup_{\ell=0}^{j^*} \mathcal{N}^{(\ell)},$$

now it is easy to check that $\mathcal{M}'$ is an $(\varepsilon, \alpha, \pi h)$-semiregular matching. Using (5.17) together with (B) and (c), we see that **(C2)** holds for $\mathcal{M}'$.

In order to see **(C1)**, we calculate

$$\begin{aligned}
|V(\mathcal{M}) \setminus V(\mathcal{M}')| &\leqslant \sum_{(A,B)\in\mathcal{M}\setminus\mathcal{L}} |A' \cup B'| \ + \sum_{(A,B)\in\mathcal{L}} \sum_{\ell=1}^{j^*} \Big(|A \cap V_1(\mathcal{N}^{(\ell-1)})| - |B \cap V_2(\mathcal{N}^{(\ell)})|\Big) \\
&\leqslant \frac{\varepsilon}{2} \cdot \sum_{(A,B)\in\mathcal{M}\setminus\mathcal{L}} |A \cup B| \ + \sum_{\ell=1}^{j^*} \Big(|V_1(\mathcal{N}^{(\ell-1)})| - |V_2(\mathcal{N}^{(\ell)})|\Big) \\
&\overset{(D)}{\leqslant} \frac{\varepsilon}{2} n \ + \sum_{\ell=1}^{j^*} \frac{\varepsilon}{2} \cdot |V_2(\mathcal{M}^{(\ell)})| \\
&\leqslant \varepsilon n \, .
\end{aligned} \tag{5.18}$$

Using the fact that $V_2(\mathcal{N}^{(0)}) \subseteq V(\mathcal{M}') \setminus V(\mathcal{M})$ the last calculation also implies that

$$\begin{aligned}
|V(\mathcal{M}')| - |V(\mathcal{M})| &\geqslant |V_2(\mathcal{N}^{(0)})| - |V(\mathcal{M}) \setminus V(\mathcal{M}')| \\
&\overset{(D)}{\geqslant} \tau' n - \varepsilon n \\
&> \frac{\tau'}{2} n \, ,
\end{aligned}$$

since $\varepsilon < \alpha \leqslant \tau'/2$ by assumption. □

Iterating Lemma 5.9 we prove the main result of the section.

Lemma 5.10. *For every $\Omega \in \mathbb{N}$, $\rho \in (0, 1/\Omega)$ there exists a number $\beta > 0$ such that for every $\varepsilon \in (0, \beta)$, there are $\varepsilon', \pi > 0$ such that for each $\gamma > 0$ there exists $k_0 \in \mathbb{N}$ such that the following holds for every $k > k_0$ and $c \in (\gamma k, k/2)$.*

Let G be a graph of order n, with $\deg^{\max}(G) \leqslant \Omega k$. Let $\mathcal{D}$ be a $(\rho k, \rho)$-dense cover of G, and let $\mathcal{M}$ be an (ε', ρ, c)-semiregular matching that is absorbed by $\mathcal{D}$. Let $\mathcal{C}$ be a c-ensemble in G outside $V(\mathcal{M})$. Let $Y \subseteq V(G) \setminus (V(\mathcal{M}) \cup \bigcup \mathcal{C})$. Assume that for each $(U, W; F) \in \mathcal{D}$, and for each $C \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$ we have that

$$U \cap C \in \{\emptyset, C\} . \tag{5.19}$$

Then there exists an $(\varepsilon, \beta, \pi c)$-semiregular matching $\mathcal{M}'$ such that

(i) $|V(\mathcal{M}) \setminus V(\mathcal{M}')| \leqslant \varepsilon n$,

(ii) for each $(T, Q) \in \mathcal{M}'$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$, and

(iii) $\mathcal{M}'$ can be partitioned into $\mathcal{M}_1$ and $\mathcal{M}_2$ so that

$$e\Big((\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus V_1(\mathcal{M}_1)\ ,\ (Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_2)\Big)\ <\ \rho k n\ .$$

Proof. Let Ω and ρ be given. Let $\tau' := \tau'_{\triangleright \mathrm{L5.9}}$ be the output given by Lemma 5.9 for input parameters $\Omega_{\triangleright \mathrm{L5.9}} := \Omega$ and $\tau_{\triangleright \mathrm{L5.9}} := \rho/2$.

Set $\rho^{(0)} := \rho$, set $L := \lceil 2/\tau' \rceil + 1$, and for $\ell \in [L]$, inductively define $\rho^{(\ell)}$ to be the output $\alpha_{\triangleright \mathrm{L5.9}}$ given by Lemma 5.9 for the further input parameter $\rho_{\triangleright \mathrm{L5.9}} := \rho^{(\ell-1)}$ (keeping $\Omega_{\triangleright \mathrm{L5.9}} = \Omega$ and $\tau_{\triangleright \mathrm{L5.9}} = \rho/2$ fixed). Then $\rho^{(\ell+1)} \leqslant \rho^{(\ell)}$ for all ℓ. Set $\beta := \rho^{(L)}$.

Given $\varepsilon < \beta$ we set $\varepsilon^{(\ell)} := (\varepsilon/2)^{3^{L-\ell}}$ for $\ell \in [L] \cup \{0\}$, and set $\varepsilon' := \varepsilon^{(0)}$. Clearly,

$$\sum_{\ell=0}^{L} \varepsilon^{(\ell)}\ \leqslant\ \varepsilon. \tag{5.20}$$

Now, for $\ell + 1 \in [L]$, let $\pi^{(\ell)} := \pi_{\triangleright \mathrm{L5.9}}$ be given by Lemma 5.9 for input parameters $\Omega_{\triangleright \mathrm{L5.9}} := \Omega$, $\tau_{\triangleright \mathrm{L5.9}} := \rho/2$, $\rho_{\triangleright \mathrm{L5.9}} := \rho^{(\ell)}$ and $\varepsilon_{\triangleright \mathrm{L5.9}} := \varepsilon^{(\ell+1)}$. For $\ell \in [L] \cup \{0\}$, set $\Pi^{(\ell)} := \frac{\rho}{2\Omega} \prod_{j=0}^{\ell-1} \pi^{(j)}$. Let $\pi := \Pi^{(L)}$.

Given γ, let k_0 be the maximum of the lower bounds $k_{0_{\triangleright \mathrm{L5.9}}}$ given by Lemma 5.9 for input parameters $\Omega_{\triangleright \mathrm{L5.9}} := \Omega$, $\tau_{\triangleright \mathrm{L5.9}} := \rho/2$, $\rho_{\triangleright \mathrm{L5.9}} := \rho^{(\ell-1)}$, $\varepsilon_{\triangleright \mathrm{L5.9}} := \varepsilon^{(\ell)}$, $\gamma_{\triangleright \mathrm{L5.9}} := \gamma \Pi^{(\ell)}$, for $\ell \in [L]$.

Suppose now we are given G, $\mathcal{D}$, $\mathcal{C}$, Y and $\mathcal{M}$. Suppose further that $c > \gamma k > \gamma k_0$. Let $\ell \in \{0, 1, \ldots, L\}$ be maximal such that there is a matching $\mathcal{M}^{(\ell)}$ with the following properties:

(a) $\mathcal{M}^{(\ell)}$ is an $(\varepsilon^{(\ell)}, \rho^{(\ell)}, \Pi^{(\ell)} c)$-semiregular matching,

(b) $|V(\mathcal{M}^{(\ell)})| \geqslant \ell \cdot \frac{\tau'}{2} n$,

(c) $|V(\mathcal{M}) \setminus V(\mathcal{M}^{(\ell)})| \leqslant \sum_{i=0}^{\ell} \varepsilon^{(\ell)} n$, and

(d) for each $(T,Q) \in \mathcal{M}^{(\ell)}$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$.

Observe that such a number ℓ exists, as for $\ell = 0$ we may take $\mathcal{M}^{(0)} = \mathcal{M}$. Also note that $\ell \leqslant 2/\tau' < L$ because of (b).

We now apply Lemma 5.9 with input parameters $\Omega_{\triangleright \mathrm{L}5.9} := \Omega$, $\tau_{\triangleright \mathrm{L}5.9} := \rho/2$, $\rho_{\triangleright \mathrm{L}5.9} := \rho^{(\ell)}$, $\varepsilon_{\triangleright \mathrm{L}5.9} := \varepsilon^{(\ell+1)} < \beta \leqslant \rho^{(\ell+1)} = \alpha_{\triangleright \mathrm{L}5.9}$, $\gamma_{\triangleright \mathrm{L}5.9} := \gamma \Pi^{(\ell)}$ to the graph G with the $(\rho^{(\ell)}k, \rho^{(\ell)})$-dense cover $\mathcal{D}$, the $(\varepsilon^{(\ell)}, \rho^{(\ell)}, \Pi^{(\ell)}c)$-semiregular matching $\mathcal{M}^{(\ell)}$, the set

$$\tilde{Y} := (Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}^{(\ell)}),$$

and the $(\Pi^{(\ell)}c)$-ensemble

$$\tilde{\mathcal{C}} := \left\{ C \setminus V(\mathcal{M}^{(\ell)}) \ : \ C \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C},\ |C \setminus V_1(\mathcal{M}^{(\ell)})| \geqslant \Pi^{(\ell)}c \right\}.$$

Lemma 5.9 yields a semiregular matching which either corresponds to $\mathcal{M}''$ as in Assertion (I) or to $\mathcal{M}'$ as in Assertion (II). Note that in the latter case, the matching $\mathcal{M}'$ actually constitutes an $(\varepsilon^{(\ell+1)}, \rho^{(\ell+1)}, \Pi^{(\ell+1)}c)$-semiregular matching $\mathcal{M}^{(\ell+1)}$ fulfilling all the above properties for $\ell+1 \leqslant L$. In fact, (b) and (c) hold for $\mathcal{M}^{(\ell+1)}$ because of (**C1**), and it is not difficult to deduce (d) from (**C2**) and from (d) for ℓ. But this contradicts the choice of ℓ. We conclude that we obtained a semiregular matching $\mathcal{M}'' \subseteq \mathcal{M}^{(\ell)}$ as in Assertion (I) of Lemma 5.9.

Thus, in other words, $\mathcal{M}^{(\ell)}$ can be partitioned into $\mathcal{M}_1$ and $\mathcal{M}_2$ so that

$$e\Big(\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}_2)\ ,\ \tilde{Y} \cup V_2(\mathcal{M}_1)\Big) \ < \ \tau_{\triangleright \mathrm{L}5.9} kn \ = \ \rho kn/2. \tag{5.21}$$

Set $\mathcal{M}' := \mathcal{M}^{(\ell)}$. Then $\mathcal{M}'$ is $(\varepsilon, \beta, \pi c)$-semiregular by (a). Note that Assertion (i) of the lemma holds by (5.20) and by (c). Assertion (ii) holds because of (d).

Since

$$(Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_2) \ \subseteq \ \tilde{Y} \cup V_2(\mathcal{M}_1),$$

and because of (5.21) we know that in order to prove Assertion (iii) it suffices to show that the set

$$\begin{aligned} X \ &:= ((\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus V_1(\mathcal{M}_1)) \setminus (\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}_2)) \\ &= (\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus (\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}^{(\ell)})) \end{aligned}$$

sends at most $\rho kn/2$ edges to the rest of the graph. For this, it would be enough to see that $|X| \leqslant \frac{\rho}{2\Omega}n$, as by assumption, G has maximum degree Ωk.

To this end, note that by assumption, $|\mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}| \leqslant \frac{n}{c}$. Further, the definition of $\tilde{\mathcal{C}}$ implies that for each $A \in \mathcal{C} \cup \mathcal{V}_1(\mathcal{M})$ we have that $|A \setminus \big(\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}^{(\ell)})\big)| \leqslant \Pi^{(\ell)}c$. Combining these two observations, we obtain that

$$|X| < \Pi^{(\ell)} n \leqslant \frac{\rho}{2\Omega} n\ ,$$

as desired.

□

6 Rough structure of LKS graphs

In this section we give a structural result for graphs $G \in \mathbf{LKSsmall}(n,k,\eta)$, stated in Lemma 6.1. Similar structural results were essential also for proving Conjecture 1.2 in the dense setting in [AKS95, PS12]. There, a certain matching structure was proved to exist in the cluster graph of the host graph. This matching structure then allowed to embed a given tree into the host graph.

Naturally, in our possibly sparse setting the sparse decomposition ∇ of G will enter the picture (instead of just the cluster graph of G). There is an important subtlety though: we need to "re-regularize" the cluster graph $\mathbf{G}_{\mathrm{reg}}$ of ∇. The necessity of this step arises from the ambiguity of the sparse decomposition ∇ given by Lemma 4.14, see Remark 4.15. Consequently, the cluster graph $\mathbf{G}_{\mathrm{reg}}$ given by a sparse decomposition $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of G might not be suitable for locating a matching structure in analogue to the dense setting. In this case, we have to find another regularization of parts of G, partially based on G_{reg}. Lemma 5.10 is the main tool to this end. The re-regularization is captured by the semiregular matchings $\mathcal{M}_A$ and $\mathcal{M}_B$.

Let us note that this step is one of the biggest differences between our approach and the announced solution of the Erdős-Sós Conjecture by Ajtai, Komlós, Simonovits and Szemerédi. In other words, the nature of the graphs arising in the Erdős-Sós Conjecture allows a less careful approach with respect to regularization, still yielding a structure suitable for embedding trees. We discuss the necessity of this step in further detail in Section 6.2, after proving the main result of this section, Lemma 6.1, in Section 6.1.

6.1 Finding the structure

We now introduce some notation we need in order to state Lemma 6.1. Suppose that G is a graph with a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition

$$\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$$

with respect to $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$. Suppose further that $\mathcal{M}_A, \mathcal{M}_B$ are $(\varepsilon', d, \gamma k)$-semiregular matchings in $G_{\mathcal{D}}$. We then define the triple $(\mathbb{XA}, \mathbb{XB}, \mathbb{XC}) = (\mathbb{XA}, \mathbb{XB}, \mathbb{XC})(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$ by setting

$$\begin{aligned}
\mathbb{XA} &:= \mathbb{L}_{\eta,k}(G) \setminus V(\mathcal{M}_B)\ , \\
\mathbb{XB} &:= \left\{ v \in V(\mathcal{M}_B) \cap \mathbb{L}_{\eta,k}(G)\ :\ \widehat{\deg}(v) < (1+\eta)\frac{k}{2} \right\}\ , \\
\mathbb{XC} &:= \mathbb{L}_{\eta,k}(G) \setminus (\mathbb{XA} \cup \mathbb{XB})\ ,
\end{aligned}$$

where $\widehat{\deg}(v)$ on the second line is defined by

$$\widehat{\deg}(v) := \deg_G\big(v, \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B))\big)\ . \tag{6.1}$$

Clearly, $\{\mathbb{XA}, \mathbb{XB}, \mathbb{XC}\}$ is a partition of $\mathbb{L}_{\eta,k}(G)$.

We now give the main and only lemma of this section, a structural result for graphs from $\mathbf{LKSsmall}(n,k,\eta)$.

Lemma 6.1. *For every $\eta > 0, \Omega > 0, \gamma \in (0, \eta/3)$ there is $\beta > 0$ so that for every $\varepsilon \in (0, \frac{\gamma^2\eta}{12})$ there exist $\varepsilon', \pi > 0$ such that for every $\nu > 0$ there exists $k_0 \in \mathbb{N}$ such that for every Ω^* with $\Omega^* < \Omega$ and every k with $k > k_0$ the following holds.*

Suppose $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ is a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition of a graph $G \in \mathbf{LKSsmall}(n, k, \eta)$ with respect to $S := \mathbb{S}_{\eta,k}(G)$ and $L := \mathbb{L}_{\eta,k}(G)$ which captures all but at most $\eta kn/6$ edges of G. Let $\mathfrak{c}$ be the size of the clusters $\mathbf{V}$.[13] *Write*

$$S^0 := S \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A}) \ . \tag{6.2}$$

Then $G_{\mathcal{D}}$ contains two $(\varepsilon, \beta, \pi\mathfrak{c})$-semiregular matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ such that for the triple $(\mathbb{XA}, \mathbb{XB}, \mathbb{XC}) := (\mathbb{XA}, \mathbb{XB}, \mathbb{XC})(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$ we have

(a) $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$,

(b) $V_1(\mathcal{M}_B) \subseteq S^0$,

(c) for each $(T, Q) \in \mathcal{M}_A \cup \mathcal{M}_B$, there is a dense spot $(A_D, B_D; E_D) \in \mathcal{D}$ with $T \subseteq A_D$, $Q \subseteq B_D$, and furthermore, either $T \subseteq S$ or $T \subseteq L$, and $Q \subseteq S$ or $Q \subseteq L$,

(d) for each $X_1 \in \mathcal{V}_1(\mathcal{M}_A \cup \mathcal{M}_B)$ there exists a cluster $C_1 \in \mathbf{V}$ such that $X_1 \subseteq C_1$, and for each $X_2 \in \mathcal{V}_2(\mathcal{M}_A \cup \mathcal{M}_B)$ there exists $C_2 \in \mathbf{V} \cup \{L \cap \mathfrak{A}\}$ such that $X_2 \subseteq C_2$,

(e) $e_{G_\nabla}\big(\mathbb{XA}, S^0 \setminus V(\mathcal{M}_A)\big) \leqslant \gamma kn$,

(f) $e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \varepsilon\Omega^* kn$,

(g) for the semiregular matching $\mathcal{N}_{\mathfrak{A}} := \{(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B \ : \ (X \cup Y) \cap \mathfrak{A} \neq \emptyset\}$ we have $e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})\big) \leqslant \varepsilon\Omega^* kn$,

(h) for $\mathcal{M}_{\mathrm{good}} := \{(A, B) \in \mathcal{M}_A \ : \ A \cup B \subseteq \mathbb{XA}\}$ we have that each $\mathcal{M}_{\mathrm{good}}$-edge is an edge of $\mathbf{G}_{\mathrm{reg}}$, and at least one of the following conditions holds

(K1) $2e_G(\mathbb{XA}) + e_G(\mathbb{XA}, \mathbb{XB}) \geqslant \eta kn/3$,

(K2) $|V(\mathcal{M}_{\mathrm{good}})| \geqslant \eta n/3$.

Remark 6.2. *In some sense, property* (h) *is the most important part of Lemma 6.1. Note that the assertion* **(K2)** *implies a quantitatively weaker version of* **(K1)**. *Indeed, consider $(C, D) \in \mathcal{M}_A$. An average vertex $v \in C$ sends at least $\beta \cdot \pi\mathfrak{c} \geqslant \beta \cdot \pi\nu k$ edges to D. Thus, if $|V(\mathcal{M}_{\mathrm{good}})| \geqslant \eta n/3$ then $\mathcal{M}_{\mathrm{good}}$ induces at least $(\eta n/6) \cdot \beta \cdot \pi\nu k = \Theta(kn)$ edges in $\mathbb{XA}$. Such a bound, however, would be insufficient for our purposes as later $\eta \gg \pi, \nu$.*

[13] The number $\mathfrak{c}$ is irrelevant when $\mathbf{V} = \emptyset$. In particular, note that in that case we necessarily have $\mathcal{M}_A = \mathcal{M}_B = \emptyset$ for the semiregular matchings given by the lemma.

Proof of Lemma 6.1. The idea of the proof is to first obtain some information about the structure of the graph $\mathbf{G}_{\text{reg}}$ with the help of the Gallai-Edmonds Matching Theorem (Theorem 2.4). Then this rough structure is refined by Lemma 5.10 to yield the assertions of the lemma.

Let us begin with setting the parameters. Let $\beta := \beta_{\triangleright \text{L5.10}}$ be given by Lemma 5.10 for input parameters $\Omega_{\triangleright \text{L5.10}} := \Omega$, $\rho_{\triangleright \text{L5.10}} := \gamma^2$, and let ε' and π be given by Lemma 5.10 for further input parameter $\varepsilon_{\triangleright \text{L5.10}} := \varepsilon$. Last, let k_0 be given by Lemma 5.10 with the above parameters and $\gamma_{\triangleright \text{L5.10}} := \nu$.

Without loss of generality we assume that $\varepsilon' \leqslant \varepsilon$ and $\beta < \gamma^2$. We write $\mathbf{S} := \{C \in \mathbf{V} : C \subseteq S\}$ and $\mathbf{L} := \{C \in \mathbf{V} : C \subseteq L\}$. Further, let $\mathbf{S}^0 := \{C \in \mathbf{S} : C \subseteq S^0\}$.

Let $\mathbf{Q}$ be a separator and N_0 a matching given by Theorem 2.4 applied to the graph $\mathbf{G}_{\text{reg}}$. We will presume that the pair $(\mathbf{Q}, N_0)$ is chosen among all the possible choices so that the number of vertices of $\mathbf{S}^0$ that are isolated in $\mathbf{G}_{\text{reg}} - \mathbf{Q}$ and are not covered by N_0 is minimized. Let $\mathbf{S}^{\text{I}}$ denote the set of vertices in $\mathbf{S}^0$ that are isolated in $\mathbf{G}_{\text{reg}} - \mathbf{Q}$. Recall that the components of $\mathbf{G}_{\text{reg}} - \mathbf{Q}$ are factor critical.

Define $\mathbf{S}^{\text{R}} \subseteq V(\mathbf{G}_{\text{reg}})$ as a minimal set such that

- $\mathbf{S}^{\text{I}} \setminus V(N_0) \subseteq \mathbf{S}^{\text{R}}$, and
- if $C \in \mathbf{S}$ and there is an edge $DZ \in E(\mathbf{G}_{\text{reg}})$ with $Z \in \mathbf{S}^{\text{R}}$, $D \in \mathbf{Q}$, $CD \in N_0$ then $C \in \mathbf{S}^{\text{R}}$.

Then each vertex from $\mathbf{S}^{\text{R}}$ is reachable from $\mathbf{S}^{\text{I}} \setminus V(N_0)$ by a path in $\mathbf{G}_{\text{reg}}$ that alternates between $\mathbf{S}^{\text{R}}$ and $\mathbf{Q}$, and has every second edge in N_0. Also note that for all $CD \in N_0$ with $C \in \mathbf{Q}$ and $D \in \mathbf{S}^0 \setminus \mathbf{S}^{\text{R}}$ we have

$$\deg_{\mathbf{G}_{\text{reg}}}(C, \mathbf{S}^{\text{R}}) = 0 \; . \tag{6.3}$$

Let us show another property of $\mathbf{S}^{\text{R}}$.

Claim 6.1.1. $\mathbf{S}^{\text{R}} \subseteq \mathbf{S}^{\text{I}} \subseteq \mathbf{S}^{\text{R}} \cup V(N_0)$. In particular, $\mathbf{S}^{\text{R}} \subseteq \mathbf{S}^0$.

Proof of Claim 6.1.1. By the definition of $\mathbf{S}^{\text{R}}$, we only need to show that $\mathbf{S}^{\text{R}} \subseteq \mathbf{S}^{\text{I}}$. So suppose there is a vertex $C \in \mathbf{S}^{\text{R}} \setminus \mathbf{S}^{\text{I}}$. By the definition of $\mathbf{S}^{\text{R}}$ there is a non-trivial path R going from C to $\mathbf{S}^{\text{I}} \setminus V(N_0)$, that alternates between $\mathbf{S}^{\text{R}}$ and $\mathbf{Q}$, and has every second edge in N_0. Then, the matching $N_0' := N_0 \triangle E(R)$ covers more vertices of $\mathbf{S}^{\text{I}}$ than N_0 does. Further, it is straightforward to check that the separator $\mathbf{Q}$ together with the matching N_0' satisfies the assertions of Theorem 2.4. This is a contradiction, as desired. $\square$

Using a very similar alternating path argument we see the following.

Claim 6.1.2. If $CD \in N_0$ with $C \in \mathbf{Q}$ and $D \notin \mathbf{S}^{\text{I}}$ then $\deg_{\mathbf{G}_{\text{reg}}}(C, \mathbf{S}^{\text{R}}) = 0$.

Using the factor-criticality of the components of $\mathbf{G}_{\text{reg}} - \mathbf{Q}$ we extend N_0 to a matching N_1 as follows. For each component K of $\mathbf{G}_{\text{reg}} - \mathbf{Q}$ which meets $V(N_0)$, we add a perfect matching of $K - V(N_0)$. Furthermore, for each non-singleton component K of $\mathbf{G}_{\text{reg}} - \mathbf{Q}$ which does not meet $V(N_0)$, we add a matching which meets all but exactly one vertex of $\mathbf{L} \cap V(K)$. This is possible

as by the definition of the class $\mathbf{LKSsmall}(n,k,\eta)$ we have that $\mathbf{G}_{\mathrm{reg}}-\mathbf{L}$ is independent, and so $\mathbf{L}\cap V(K)\neq\emptyset$. This choice of N_1 guarantees that

$$e_{\mathbf{G}_{\mathrm{reg}}}(\mathbf{V}\setminus V(N_1))=0\;. \tag{6.4}$$

We set

$$M:=\left\{AB\in N_0\;:\;A\in\mathbf{S}^{\mathrm{R}},B\in\mathbf{Q}\right\}\;.$$

We have that

$$e_{\mathbf{G}_{\mathrm{reg}}}\left(\mathbf{V}\setminus V(N_1),V(M)\cap\mathbf{S}^{\mathrm{R}}\right)=0\;. \tag{6.5}$$

As $\mathbf{S}$ is an independent set in $\mathbf{G}_{\mathrm{reg}}$, we have that

$$\mathbf{Q}_M:=V(M)\cap\mathbf{Q}\subseteq\mathbf{L}\;. \tag{6.6}$$

The matching M in $\mathbf{G}_{\mathrm{reg}}$ corresponds to an $(\varepsilon',\gamma^2,\mathfrak{c})$-semiregular matching $\mathcal{M}$ in the underlying graph G_{reg}, with $V_2(\mathcal{M})\subseteq\bigcup\mathbf{Q}$ (recall that semiregular matchings have orientations on their edges). Likewise, we define $\mathcal{N}_1$ as the $(\varepsilon',\gamma^2,\mathfrak{c})$-regular matching corresponding to N_1. The $\mathcal{N}_1$-edges are oriented so that $V_1(\mathcal{N}_1)\cap\bigcup\mathbf{Q}=\emptyset$; this condition does not specify orientations of all the $\mathcal{N}_1$-edges and we orient the remaining ones in an arbitrary fashion. We write $S^{\mathrm{R}}:=\bigcup\mathbf{S}^{\mathrm{R}}$.

Claim 6.1.3. $e_{G_\nabla}\left(L\setminus(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\right)=0$.

Proof of Claim 6.1.3. We start by showing that for every cluster $C\in\mathbf{L}\setminus V(M)$ we have

$$\deg_{\mathbf{G}_{\mathrm{reg}}}(C,\mathbf{S}^{\mathrm{R}})=0\;. \tag{6.7}$$

First, if $C\notin\mathbf{Q}$, then (6.7) is true since $\mathbf{S}^{\mathrm{R}}\subseteq\mathbf{S}^{\mathrm{I}}$ by Claim 6.1.1. So suppose that $C\in\mathbf{Q}$, and let $D\in V(\mathbf{G}_{\mathrm{reg}})$ be such that $DC\in N_0$. Now if $D\notin\mathbf{S}^{\mathrm{I}}$ then (6.7) follows from Claim 6.1.2. On the other hand, suppose $D\in\mathbf{S}^{\mathrm{I}}\subseteq\mathbf{S}^{0}$. As $C\notin V(M)$, we know that $D\notin\mathbf{S}^{\mathrm{R}}$, and thus, (6.7) follows from (6.3).

Now, by (6.7), G_{reg} has no edges between $L\setminus(\mathfrak{A}\cup V(\mathcal{M}))$ and S^{R}. Also, no such edges can be in G_{exp} or incident with $\mathfrak{A}$, since $\mathbf{S}^{\mathrm{R}}\subseteq\mathbf{S}^{0}$ by Claim 6.1.1. Finally, since $G\in\mathbf{LKSsmall}(n,k,\eta)$, there are no edges between $\mathbf{\Psi}$ and S. This proves the claim. □

We prepare ourselves for an application of Lemma 5.10. The numerical parameters of the lemma are $\Omega_{\triangleright\mathrm{L}5.10},\rho_{\triangleright\mathrm{L}5.10},\varepsilon_{\triangleright\mathrm{L}5.10}$ and $\gamma_{\triangleright\mathrm{L}5.10}$ as above. The input objects for the lemma are the graph $G_{\mathcal{D}}$ of order $n'\leqslant n$, the collection of $(\gamma k,\gamma)$-dense spots $\mathcal{D}$, the matching $\mathcal{M}$, the (νk)-ensemble $\mathcal{C}_{\triangleright\mathrm{L}5.10}:=\mathbf{S}^{\mathrm{R}}\setminus V(N_1)$, and the set $Y_{\triangleright\mathrm{L}5.10}:=L\cap\mathfrak{A}$. Note that Definition 4.7, item 5, implies that $\mathcal{D}$ absorbes $\mathcal{M}$. Further, (5.19) is satisfied by Definition 4.7, item 6.

The output of Lemma 5.10 is an $(\varepsilon,\beta,\pi\mathfrak{c})$-semiregular matching $\mathcal{M}'$ with the following properties.

(I) $|V(\mathcal{M})\setminus V(\mathcal{M}')|<\varepsilon n'\leqslant\varepsilon n$.

(II) For each $(T,U) \in \mathcal{M}'$ there are sets $C \in \mathbf{S}^{\mathrm{R}}$ and $D = (A_D, B_D; E_D) \in \mathcal{D}$ such that $T \subseteq C \cap A_D$ and $U \subseteq ((L \cap \mathfrak{A}) \cup \bigcup \mathbf{Q}_M) \cap B_D$.

(Indeed, to see this we use that $\mathcal{V}_1(\mathcal{M}) \subseteq \mathbf{S}^{\mathrm{R}}$ and that $V_2(\mathcal{M}) \subseteq \bigcup \mathbf{Q}_M$ by the definition of $\mathcal{M}$.)

(III) There is a partition of $\mathcal{M}'$ into $\mathcal{M}_1$ and $\mathcal{M}_B$ such that

$$e_{G_{\mathcal{D}}}\left(\left(\left(S^{\mathrm{R}} \setminus V(\mathcal{N}_1)\right) \cup V_1(\mathcal{M}) \right) \setminus V_1(\mathcal{M}_1) \, , \, ((L \cap \mathfrak{A}) \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_B) \right) < \gamma k n' \, .$$

We claim that also

(IV) $V(\mathcal{M}') \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$.

Indeed, let $(T,U) \in \mathcal{M}'$ be arbitrary. Then by (II) there is $C \in \mathbf{S}^{\mathrm{R}}$ such that $T \subseteq C$. By Claim 6.1.1, C is a singleton component of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$. In particular, if C is covered by N_1 then $C \in V(M)$. It follows that $T \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$. In a similar spirit, the easy fact that $(Y \cup \bigcup \mathbf{Q}_M) \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$ together with (II) gives $U \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$. This establishes (IV).

Observe that (II) implies that $V_1(\mathcal{M}') \subseteq S^{\mathrm{R}}$, and so, by Claim 6.1.1 we know that

$$V_1(\mathcal{M}_B) \subseteq S^{\mathrm{R}} \subseteq \bigcup \mathbf{S}^{\mathrm{I}} \subseteq S^0. \tag{6.8}$$

Set

$$\mathcal{M}_A := (\mathcal{N}_1 \setminus \mathcal{M}) \cup \mathcal{M}_1 \, . \tag{6.9}$$

Then $\mathcal{M}_A$ is an $(\varepsilon, \beta, \pi\mathfrak{c})$-semiregular matching. Note that from now on, the sets $\mathbb{XA}, \mathbb{XB}$ and $\mathbb{XC}$ are defined. The situtation is illustrated in Figure 6.1. By (IV), we have $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$, as required for Lemma 6.1(a). Lemma 6.1(b) follows from (6.8). Observe that by (II), also Lemma 6.1(c) and Lemma 6.1(d) are satisfied.

We now turn to Lemma 6.1(e). First we prove some auxiliary statements.

Claim 6.1.4. We have $\mathbf{S}^0 \setminus V(N_1 \setminus M) \subseteq \mathbf{S}^{\mathrm{R}}$.

Proof of Claim 6.1.4. Let $C \in \mathbf{S}^0 \setminus V(N_1 \setminus M)$. Note that if $C \notin \mathbf{S}^{\mathrm{I}}$, then $C \in V(N_1)$. On the other hand, if $C \in \mathbf{S}^{\mathrm{I}}$, then we use Claim 6.1.1 to see that $C \in \mathbf{S}^{\mathrm{R}} \cup V(N_1)$. We deduce that in either case $C \in \mathbf{S}^{\mathrm{R}} \cup V(N_1)$. The choice of C implies that thus $C \in \mathbf{S}^{\mathrm{R}} \cup V(M)$. Now, if $C \in V(M)$, then $C \in \mathbf{S}^{\mathrm{R}}$ by (6.6) and by the definition of M. Thus $C \in \mathbf{S}^{\mathrm{R}}$ as desired. $\square$

It will be convenient to work with a set $\bar{S}^0 \subseteq S^0$, $\bar{S}^0 := (S \cap \bigcup \mathbf{V}) \setminus V(G_{\mathrm{exp}}) = \bigcup \mathbf{S}^0$. Note that $\bar{S}^0$ is essentially the same as S^0; the vertices in $S^0 \setminus \bar{S}^0$ are isolated in G_{∇} and thus have very little effect on our considerations.

By Claim 6.1.4, we have

$$\bar{S}^0 \setminus V(\mathcal{M}_A) \subseteq \left(\bigcup \mathbf{S}^0 \setminus V(N_1 \setminus M) \right) \setminus V(\mathcal{M}_A) \subseteq S^{\mathrm{R}} \setminus V(\mathcal{M}_A). \tag{6.10}$$

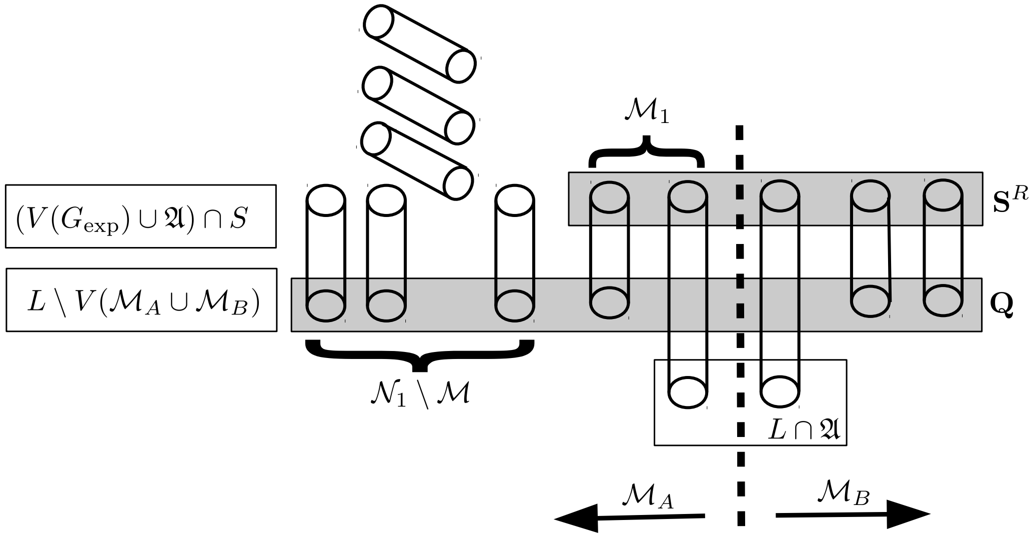

Figure 6.1: The situation in G after applying Lemma 5.10. The dotted line illustrates the separation as in (III).

As every edge incident to $S^0 \setminus \bar{S}^0$ is uncaptured, we see that

$$\begin{aligned} E_{G_\nabla}\big(\mathbb{XA} \cap \mathfrak{A}, S^0 \setminus V(\mathcal{M}_A)\big) &\subseteq E_{G_\mathcal{D}}\big((L \cap \mathfrak{A}) \setminus V(\mathcal{M}_B), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.10))} \quad &\subseteq E_{G_\mathcal{D}}\big(\,(L \cap \mathfrak{A}) \setminus V(\mathcal{M}_B)\,,\ S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\,\big). \end{aligned} \tag{6.11}$$

We claim that furthermore

$$E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, S^0 \setminus V(\mathcal{M}_A)\big) \subseteq E_{G_\mathcal{D}}\left(((L \cap \mathfrak{A}) \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_B), S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\,\right). \tag{6.12}$$

Before proving (6.12), let us see that it implies Lemma 6.1(e). As $G \in \mathbf{LKSsmall}(n,k,\eta)$, there are no edges between $\mathbf{\Psi}$ and S. That means that any captured edge from $\mathbb{XA}$ to $S^0 \setminus V(\mathcal{M}_A)$ must start in $\mathfrak{A}$ or in $\bigcup \mathbf{V}$. Thus Lemma 6.1(e) follows by plugging (III) into (6.11) and (6.12).

Let us now prove (6.12). First, observe that by the definition of $\mathbb{XA}$ and by the definition of $\mathcal{M}$ (and M) we have

$$\mathbb{XA} \cap \bigcup \mathbf{V} \subseteq (V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B)) \cup (L \setminus (\mathfrak{A} \cup V(\mathcal{M})))\,. \tag{6.13}$$

Further, by applying (6.10) and Claim 6.1.3 we get

$$E_{G_{\mathrm{reg}}}\big(L \setminus (\mathfrak{A} \cup V(\mathcal{M})), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) = \emptyset\,. \tag{6.14}$$

Therefore, we obtain

$$\begin{aligned} E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, S^0 \setminus V(\mathcal{M}_A)\big) &\subseteq E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.13))} \quad &\subseteq E_{G_{\mathrm{reg}}}\big(V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ &\quad \cup E_{G_{\mathrm{reg}}}\big(L \setminus (\mathfrak{A} \cup V(\mathcal{M})), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.10),\ (6.14))} \quad &\subseteq E_{G_{\mathrm{reg}}}\big(V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B), S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\big)\,, \end{aligned}$$

as needed for (6.12).

In order to prove (f) we first observe that

$$\begin{aligned} V(\mathcal{N}_1) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B) &\overset{(6.9)}{=} V(\mathcal{N}_1) \setminus V\big((\mathcal{N}_1 \setminus \mathcal{M}) \cup \mathcal{M}_1 \cup \mathcal{M}_B\big) \\ &= (V(\mathcal{N}_1) \cap V(\mathcal{M})) \setminus V(\mathcal{M}_B \cup \mathcal{M}_1) \\ &\overset{(\mathrm{III})}{=} (V(\mathcal{N}_1) \cap V(\mathcal{M})) \setminus V(\mathcal{M}') \\ &= V(\mathcal{M}) \setminus V(\mathcal{M}')\ . \end{aligned} \tag{6.15}$$

Now, we have

$$\begin{aligned} e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) &\leqslant e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{N}_1)) + \sum_{v \in V(\mathcal{N}_1) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)} \deg_{G_\nabla}(v) \\ {\scriptstyle (\text{by } (6.4) \text{ and } (6.15))} \quad &\leqslant \sum_{v \in V(\mathcal{M}) \setminus V(\mathcal{M}')} \deg_{G_\nabla}(v) \\ &\leqslant |V(\mathcal{M}) \setminus V(\mathcal{M}')| \Omega^* k \\ {\scriptstyle (\text{by (I)})} \quad &< \varepsilon \Omega^* k n\ , \end{aligned}$$

which shows (f).

Let us turn to proving (g). First, recall that we have $V(\mathcal{N}_{\mathfrak{A}}) \subseteq V(\mathcal{M}') \cup V(\mathcal{N}_1)$ (cf. 6.9). Since $V(\mathcal{N}_1) \cap \mathfrak{A} = \emptyset$ we actually have

$$V(\mathcal{N}_{\mathfrak{A}}) = V(\mathcal{N}_{\mathfrak{A}}) \cap V(\mathcal{M}')\ . \tag{6.16}$$

Using (6.16) and (II) we get

$$\begin{aligned} e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{N}_1), V(\mathcal{N}_{\mathfrak{A}})) &\leqslant e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{N}_1), V(\mathcal{M}') \cap S^{\mathrm{R}}\big) \\ {\scriptstyle (\text{by } (6.5))} \quad &\leqslant e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{N}_1), (V(\mathcal{M}') \setminus V(\mathcal{M})) \cap S^{\mathrm{R}}\big) \\ {\scriptstyle (\text{by (IV)})} \quad &\leqslant e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{N}_1), (V(\mathcal{M}') \setminus V(\mathcal{N}_1)) \cap S^{\mathrm{R}}\big) \\ &\leqslant 2 e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{N}_1)) \overset{(6.4)}{=} 0\ . \end{aligned} \tag{6.17}$$

We have

$$\begin{aligned} e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})) &\leqslant e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{N}_1), V(\mathcal{N}_{\mathfrak{A}})) \\ &\quad + e_{G_{\mathrm{reg}}}(V(\mathcal{N}_1) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(G)) \\ {\scriptstyle (\text{by } (6.17))} \quad &\leqslant 0 + |V(\mathcal{N}_1) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)| \Omega^* k \\ {\scriptstyle (\text{by } (6.15), \text{ (I)})} \quad &\leqslant \varepsilon \Omega^* k n\ , \end{aligned}$$

as needed.

We have thus shown Lemma 6.1(a)–(g). It only remains to prove Lemma 6.1(h), which we will do in the remainder of this section.

We first collect several properties of $\mathbb{XA}$ and $\mathbb{XC}$. The definitions of $\mathbb{XC}$ and S^0 give

$$|\mathbb{XC}|(1+\eta)\frac{k}{2} \leqslant e_G\big(\mathbb{XC}, S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \leqslant |S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)|(1+\eta)k \,. \tag{6.18}$$

Each $v \in \mathbb{XC}$ has neighbours in S. Thus, by 2. of Definition 2.6 we have

$$\deg_G(v) = \lceil (1+\eta)k \rceil \tag{6.19}$$

for each $v \in \mathbb{XC}$. Further, each vertex of $\mathbb{XC}$ has degree at least $(1+\eta)\frac{k}{2}$ into S, and so,

$$e_G(S, \mathbb{XC}) \geqslant |\mathbb{XC}| \left\lceil (1+\eta)\frac{k}{2} \right\rceil \,. \tag{6.20}$$

Consequently (using the elementary inequality $\lceil a \rceil - \lceil \frac{a}{2} \rceil \leqslant \frac{a}{2}$),

$$\begin{aligned} e_G\big(\mathbb{XA}, \mathbb{XC}\big) &\overset{(6.19)}{\leqslant} |\mathbb{XC}|\lceil (1+\eta)k \rceil - e_G(S, \mathbb{XC}) \\ &\overset{(6.20)}{\leqslant} |\mathbb{XC}|(1+\eta)\frac{k}{2} \qquad (6.21) \\ &\overset{(6.18)}{\leqslant} |S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)|(1+\eta)k \,. \qquad (6.22) \end{aligned}$$

Let $\mathcal{M}_{\mathrm{good}}$ be defined as in Lemma 6.1(h), that is, $\mathcal{M}_{\mathrm{good}} := \{(A,B) \in \mathcal{M}_A \,:\, A \cup B \subseteq \mathbb{XA}\}$. Note that (6.8) implies that $A \subseteq S$ for every $(A,B) \in \mathcal{M}_B$. Thus by the definition of $\mathbb{XA}$,

$$\text{if } (A,B) \in \mathcal{M}_A \cup \mathcal{M}_B \text{ with } A \cup B \subseteq L \text{ then } (A,B) \in \mathcal{M}_{\mathrm{good}}. \tag{6.23}$$

We will now show the first part of Lemma 6.1(h), that is, we show that each $\mathcal{M}_{\mathrm{good}}$-edge is an edge of $\mathbf{G}_{\mathrm{reg}}$. Indeed, by (II), we have that $V_1(\mathcal{M}_1) \subseteq S$, so as $\mathbb{XA} \cap S = \emptyset$, it follows that $\mathcal{M}_1 \cap \mathcal{M}_{\mathrm{good}} = \emptyset$. Thus $\mathcal{M}_{\mathrm{good}} \subseteq \mathcal{N}_1$. As $\mathcal{N}_1$ corresponds to a matching in $\mathbf{G}_{\mathrm{reg}}$, all is as desired.

Finally, let us assume that neither **(K1)** nor **(K2)** are fulfilled. After five preliminary observations (Claim 6.1.5–Claim 6.1.9), we will derive a contradiction from this assumption.

Claim 6.1.5. We have $|S \cap V(\mathcal{M}_A)| \leqslant |\mathbb{XA} \cap V(\mathcal{M}_A)|$.

Proof of Claim 6.1.5. To see this, recall that each $\mathcal{M}_A$-vertex $U \in \mathcal{V}(\mathcal{M}_A)$ is either contained in S, or in L. Further, if $U \subseteq S$ then its partner in $\mathcal{M}_A$ must be in L, as S is independent. Now, the claim follows after noticing that $L \cap V(\mathcal{M}_A) = \mathbb{XA} \cap V(\mathcal{M}_A)$. □

Claim 6.1.6. We have $|S \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)| + 2\eta n < |\mathbb{XA} \setminus V(\mathcal{M}_A)| + \eta n/3$.

Proof of Claim 6.1.6. As $G \in \mathbf{LKS}(n,k,\eta)$, we have $|S| + 2\eta n \leqslant |L|$. Therefore,

$$\begin{aligned} |S \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)| + 2\eta n &\leqslant |L \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)| + \sum_{\substack{(A,B) \in \mathcal{M}_A \cup \mathcal{M}_B \\ A \cup B \subseteq L}} |A \cup B| \\ &\overset{(6.23)}{=} |\mathbb{XA} \setminus V(\mathcal{M}_A)| + |V(\mathcal{M}_{\mathrm{good}})| \\ &\overset{\neg\mathbf{(K2)}}{<} |\mathbb{XA} \setminus V(\mathcal{M}_A)| + \eta n/3 \,. \end{aligned}$$

□

Claim 6.1.7. We have $e_{G_\nabla}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)<\eta kn/2$.

Proof of Claim 6.1.7. As

$$\begin{aligned}\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M}))&\subseteq((L\cap\mathfrak{A})\cup V_2(\mathcal{M}))\setminus V_2(\mathcal{M}_B)\quad\text{and}\\ S^{\mathrm{R}}\setminus V(\mathcal{M}_A)&\subseteq\big((S^{\mathrm{R}}\setminus V(\mathcal{N}_1))\cup V_1(\mathcal{M})\big)\setminus V_1(\mathcal{M}_1)\ ,\end{aligned}$$

we get from (III) that

$$e_{G_{\mathcal{D}}}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\leqslant\gamma kn\ . \tag{6.24}$$

Observe now that both sets $\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M}))$ and $S^{\mathrm{R}}\setminus V(\mathcal{M}_A)$ avoid $\mathbf{\Psi}$. Further, no edges between them belong to G_{exp}, because Claim 6.1.1 implies that $S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\subseteq S^0\subseteq V(G)\setminus V(G_{\mathrm{exp}})$. Therefore, we can pass from $G_{\mathcal{D}}$ to G_∇ in (6.24) to get

$$e_{G_\nabla}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\leqslant\gamma kn<\eta kn/2\ .$$

□

Claim 6.1.8. We have $S\setminus(S^{\mathrm{R}}\cup V(\mathcal{M}_A))\subseteq S\setminus(\bar{S}^0\cup V(\mathcal{M}_A\cup\mathcal{M}_B))$.

Proof of Claim 6.1.8. The claim follows directly from the following two inclusions.

$$S^{\mathrm{R}}\cup V(\mathcal{M}_A)\supseteq S\cap V(\mathcal{M}_A\cup\mathcal{M}_B)\ ,\text{ and} \tag{6.25}$$

$$S^{\mathrm{R}}\cup V(\mathcal{M}_A)\supseteq\bar{S}^0\ . \tag{6.26}$$

Now, (6.25) is trivial, as by (II) we have that $S^{\mathrm{R}}\supseteq S\cap V(\mathcal{M}_B)$. To see (6.26), it suffices by (6.9) to prove that $V(N_1\setminus M)\cup\mathbf{S}^{\mathrm{R}}\supseteq\mathbf{S}^0$. This is however the subject of Claim 6.1.4. □

Next, we bound $e_{G_\nabla}\big(\mathbb{XA},S\big)$.

Claim 6.1.9. We have

$$e_{G_\nabla}\big(\mathbb{XA},S\big)\leqslant|S\cap V(\mathcal{M}_A)|(1+\eta)k+|S\setminus(S^0\cup V(\mathcal{M}_A\cup\mathcal{M}_B))|(1+\eta)k+\frac{1}{2}\eta kn\ .$$

Proof of Claim 6.1.9. We have

$$\begin{aligned}e_{G_\nabla}\big(\mathbb{XA},S\big)=\ &e_{G_\nabla}\big(\mathbb{XA},S\cap V(\mathcal{M}_A)\big)\\&+e_{G_\nabla}\big(\mathbb{XA},S\setminus(S^{\mathrm{R}}\cup V(\mathcal{M}_A))\big)\\&+e_{G_\nabla}\big(\mathbb{XA}\setminus(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\\&+e_{G_\nabla}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\ .\end{aligned}$$

To bound the first term we use that the vertices in $S\cap V(\mathcal{M}_A)$ each have degree at most $(1+\eta)k$, and thus obtain $e_{G_\nabla}(\mathbb{XA},S\cap V(\mathcal{M}_A))\leqslant|S\cap V(\mathcal{M}_A)|(1+\eta)k$. To bound the second term, we again use a bound on degree of vertices of $S\setminus\big((S^{\mathrm{R}}\cup V(\mathcal{M}_A))\cup(S^0\setminus\bar{S}^0)\big)$, together with Claim 6.1.8. The third term is zero by Claim 6.1.3. The fourth term can be bounded by Claim 6.1.7. □

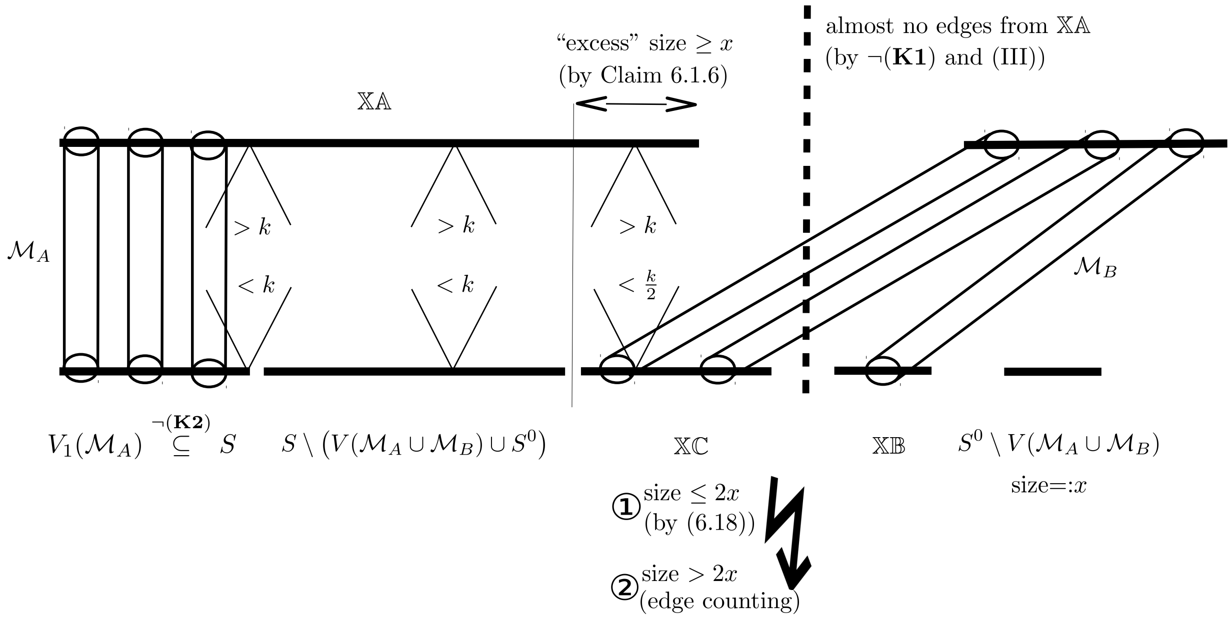

Figure 6.2: A simplified computation showing that $\neg$(**K1**), $\neg$(**K2**) leads to a contradiction. Denoting by x the size of $S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)$ we get ① $|\mathbb{XC}| \leqslant 2x$. On the other hand, each vertex of $\mathbb{XA}$ emanates $\gtrsim k$ edges which are absorbed by the sets $V_1(\mathcal{M}_A)$, $S \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup S^0)$, and $\mathbb{XC}$. The vertices of $V_1(\mathcal{M}_A)$ and $S \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup S^0)$ can absorb $\lesssim k$ edges. The vertices of $\mathbb{XC}$ receive $\lesssim \frac{k}{2}$ edges of $\mathbb{XA}$ by (6.21). This leads to ② $|\mathbb{XC}| > 2x$, doubling the size of the "excess" vertices of $\mathbb{XA}$.

A relatively short double counting below will lead to the final contradiction. The idea behind this computation is given in Figure 6.2.

$$\begin{aligned}
|\mathbb{XA}|(1+\eta)k &\leqslant \sum_{v\in\mathbb{XA}} \deg_G(v) \\
&\leqslant \sum_{v\in\mathbb{XA}} \deg_{G_\nabla}(v) + 2\big(e(G) - e(G_\nabla)\big) \\
&\leqslant 2e_{G_\nabla}(\mathbb{XA}) + e_{G_\nabla}(\mathbb{XA},\mathbb{XB}) + e_{G_\nabla}\big(\mathbb{XA},\mathbb{XC}\big) \\
&\quad + e_{G_\nabla}\big(\mathbb{XA},S\big) + \frac{\eta k n}{3} \\
\text{(by } \neg\textbf{(K1)}\text{, (6.22), C6.1.9)} \quad &\leqslant \frac{7}{6}\eta k n + \big|S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big|(1+\eta)k \\
&\quad + |S \cap V(\mathcal{M}_A)|(1+\eta)k \\
&\quad + |S \setminus (S^0 \cup V(\mathcal{M}_A \cup \mathcal{M}_B))|(1+\eta)k \qquad (6.27)\\
\text{(by C6.1.5)} \quad &\leqslant \frac{7}{6}\eta k n + |S \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)|(1+\eta)k \\
&\quad + |\mathbb{XA} \cap V(\mathcal{M}_A)|(1+\eta)k \\
\text{(by C6.1.6)} \quad &\leqslant \frac{7}{6}\eta k n + \Big(|\mathbb{XA} \setminus V(\mathcal{M}_A)| - \frac{5}{3}\eta n\Big)(1+\eta)k \\
&\quad + |\mathbb{XA} \cap V(\mathcal{M}_A)|(1+\eta)k \\
&< |\mathbb{XA}|(1+\eta)k - \frac{1}{2}\eta k n\ ,
\end{aligned}$$

a contradiction. This finishes the proof of Lemma 6.1. □

6.2 The role of Lemma 5.10 in the proof of Lemma 6.1

Let us explain the role of Lemma 5.10 in our proof of Lemma 6.1. First, let us attempt to use just the sparse decomposition ∇ to embed a tree $T \in \mathbf{trees}(k)$ in $G \in \mathbf{LKS}(n,k,\eta)$. We will eventually see that this is impossible and that we need to enhance ∇ by a semiregular matching (provided by Lemma 5.10).

We wish to find two sets $\mathbb{VA}$ and $\mathbb{VB}$ which are suitable for embedding the cut vertices W_A and W_B of a τk-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T, respectively. In this sketch we just focus on finding $\mathbb{VA}$; the ideas behind finding a set suitable set $\mathbb{VB}$ are similar.

To accommodate all the shrubs from $\mathcal{S}_A$ — which might contain up to k vertices in total — we need $\mathbb{VA}$ to have degree at least $\sum_{T^*\in\mathcal{S}_A} v(T^*)$ into a suitable set of vertices we reserve for these shrubs. (The neighbourhood of a possible image of a vertex from W_A has to allow space for its children and for everything blocked by shrubs from $\mathcal{S}_A$ embedded earlier.)

Our methods of embedding in Section 8 determine which sets we find 'suitable' for $\mathcal{S}_A$: these are the large vertices $\mathbb{L}_{\eta,k}(G)$, the vertices of the nowhere-dense graph G_{exp}, the avoiding set $\mathfrak{A}$, and any matching consisting of regular pairs. This motivates us to look for a semiregular matching $\mathcal{M}$ which covers as much as possible of the set $S^0 := \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A})$ which consists of those vertices not utilizable by any other of the methods above. As a next step one would prove that

there is a set $\mathbb{VA}$ with

$$\deg^{\min}\left(\mathbb{VA}, V(G) \setminus (S^0 \setminus V(\mathcal{M}))\right) \gtrsim k \ .$$

In the dense setting [PS12], where the structure of G is determined by $\mathbf{G}_{\mathrm{reg}}$, and where $S^0 = \mathbb{S}_{\eta,k}(G)$, such a matching $\mathcal{M}$ can be found inside $\mathbf{G}_{\mathrm{reg}}$ using the Gallai-Edmonds Matching Theorem. But here, just working with $\mathbf{G}_{\mathrm{reg}}$ is not enough for finding a suitable semiregular matching as the following example shows.

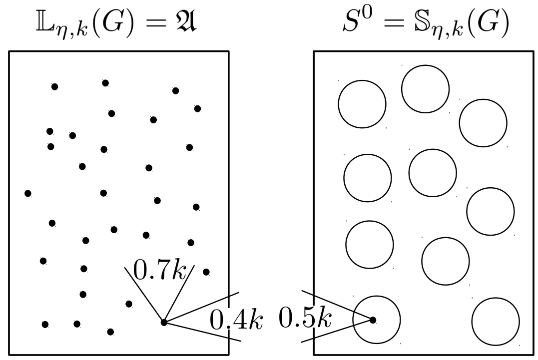

Figure 6.3: An example of a graph $G \in \mathbf{LKS}(n, k, \eta := \frac{1}{10})$ in which $\mathbf{G}_{\mathrm{reg}}$ is empty, and yet there is no candidate set for $\mathbb{VA}$ of vertices which have degrees at least k outside the set S^0.

Figure 6.3 shows a graph G with $\mathbb{L}_{\eta,k}(G) \subseteq \mathfrak{A}$, and where the vertices in $S^0 = \mathbb{S}_{\eta,k}(G)$ form clusters which do not induce any dense regular pairs. Each $\mathbb{L}_{\eta,k}(G)$-vertex sends $0.7k$ edges to $\mathbb{L}_{\eta,k}(G)$ and $0.4k$ edges to $\mathbb{S}_{\eta,k}(G)$, and each $\mathbb{S}_{\eta,k}(G)$-vertex receives $0.5k$ edges from $\mathbb{L}_{\eta,k}(G)$. The edges between $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$ are contained in $\mathcal{D}$. No vertex has degree $\gtrsim k$ outside S^0, and the cluster graph $\mathbf{G}_{\mathrm{reg}}$ contains no matching.

However in this situation we can still find a large semiregular matching $\mathcal{M}$ between $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$, by regularizing the crossing dense spots $\mathcal{D}$. (In general, obtaining a semiregular matching is of course more complicated.)

The example relates to Lemma 6.1 by setting $\mathbb{XA} := \mathbb{VA}$, and $\mathcal{M}_A := \mathcal{M}$. Indeed, (e) of Lemma 6.1 says that $\mathbb{XA}$-vertices send almost no edges to $S^0 \setminus V(\mathcal{M}_A)$, and thus (since $\mathbb{XA} \subseteq \mathbb{L}_{\eta,k}(G)$), they have degree $\gtrsim k$ outside $S^0 \setminus V(\mathcal{M}_A)$.

7 Configurations

In this section we introduce ten configurations — called ($\diamond$**1**)–($\diamond$**10**) — which may be found in a graph $G \in \mathbf{LKS}(n, k, \eta)$. We will be able to infer from the main results of this section (Lemmas 7.33–7.35) and from other structural results of this paper that each graph $G \in \mathbf{LKS}(n, k, \eta)$ contains at least one of these configurations. Lemmas 7.33–7.35 are based on the structure provided by Lemma 6.1 which itself is in a sense the most descriptive result of the structure of graphs from $\mathbf{LKS}(n, k, \eta)$. However, the structure given by Lemma 6.1 needs some burnishing. It will turn out

in Section 8 that each of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$ is suitable for the embedding of any tree from $\mathbf{trees}(k)$ as required for Theorem 1.3.

This section is organized as follows. In Section 7.1 we introduce an auxiliary notion of shadows and prove some simple properties of them. Section 7.2 introduces randomized splitting of the vertex set of an input graph. In Section 7.3 we define certain cleaned versions of the sets $\mathbb{XA}$ and $\mathbb{XB}$, and introduce other building blocks for the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$. In Section 7.4 we state some preliminary definitions and introduce the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$. In Section 7.6 we prove certain "cleaning lemmas". The main results are then stated and proved in Section 7.7. The results of Section 7.7 rely on the auxiliary lemmas of Section 7.2 and 7.6.

7.1 Shadows

We will find it convenient to work with the notion of a shadow. Given a graph H, a set $U \subseteq V(H)$, and a number ℓ we define inductively

$$\mathbf{shadow}^{(0)}_H(U,\ell) := U, \text{ and}$$
$$\mathbf{shadow}^{(i)}_H(U,\ell) := \{v \in V(H) \,:\, \deg_H(v, \mathbf{shadow}^{(i-1)}_H(U,\ell)) > \ell\} \text{ for } i \geqslant 1.$$

We abbreviate $\mathbf{shadow}^{(1)}_H(U,\ell)$ as $\mathbf{shadow}_H(U,\ell)$. Further, the graph H is omitted from the subscript if it is clear from the context. Note that the shadow of a set U might intersect U.

Below, we state two facts which bound the size of a shadow of a given set. Fact 7.1 gives a bound in general graphs of bounded maximum degree and Fact 7.2 gives a stronger bound for nowhere-dense graphs.

Fact 7.1. *Suppose H is a graph with $\deg^{\max}(H) \leqslant \Omega k$. Then for each $\alpha > 0, i \in \{0,1,\ldots\}$, and each set $U \subseteq V(H)$, we have*

$$|\mathbf{shadow}^{(i)}(U,\alpha k)| \leqslant \left(\frac{\Omega}{\alpha}\right)^i |U| \,.$$

Proof. Proceeding by induction on i it suffices to show that $|\mathbf{shadow}^{(1)}(U,\alpha k)| \leqslant \Omega|U|/\alpha$. To this end, observe that U sends out at most $\Omega k|U|$ edges while each vertex of $\mathbf{shadow}(U,\alpha k)$ receives at least αk edges from U. □

Fact 7.2. *Let $\alpha, \gamma, Q > 0$ be three numbers such that $Q \geqslant 1$ and $16Q \leqslant \frac{\alpha}{\gamma}$. Suppose that H is a $(\gamma k, \gamma)$-nowhere-dense graph, and let $U \subseteq V(H)$ with $|U| \leqslant Qk$. Then we have*

$$|\mathbf{shadow}(U,\alpha k)| \leqslant \frac{16Q^2\gamma}{\alpha}k.$$

Proof. Suppose otherwise and let $W \subseteq \mathbf{shadow}(U,\alpha k)$ be of size $|W| = \frac{16Q^2\gamma}{\alpha}k \leqslant Qk$. Then $e_H(U \cup W) \geqslant \frac{1}{2}\sum_{v\in W}\deg_H(v,U) \geqslant 8\gamma Q^2k^2$. Thus $H[U \cup W]$ has average degree at least

$$\frac{2e_H(U\cup W)}{|U|+|W|} \geqslant 8\gamma Qk \;,$$

and therefore, by a well-known fact, contains a subgraph H' of minimum degree at least $4\gamma Qk$. Taking a maximal cut (A,B) in H', it is easy to see that $H'[A,B]$ has minimum degree at least $2\gamma Qk \geqslant \gamma k$. Further, $H'[A,B]$ has density at least $\frac{|A|\cdot 2\gamma Qk}{|A||B|} \geqslant \gamma$, contradicting the fact that H is $(\gamma k, \gamma)$-nowhere-dense. $\square$

7.2 Random splitting

Suppose a graph G (together with its bounded decomposition[14]) is given. In this section we split its vertex set in several classes in a given ratio. It is important that most vertices will have their degrees split obeying approximately this ratio. The corresponding statement is given in Lemma 7.3. It will be used to split the vertices of the host graph $G = G_{\triangleright T1.3}$ according to which part of the tree $T = T_{\triangleright T1.3} \in \mathbf{trees}(k)$ they will host. More precisely, suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ is an ℓ-fine partition of T (for a suitable number ℓ). Let t_{int} and t_{end} be the total sizes of the internal and end shrubs, respectively. We then want to partition $V(G)$ into three sets $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ (which correspond to $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3$ in Lemma 7.3) in the ratio (approximately)

$$(|W_A| + |W_B|) \; : \; t_{\mathrm{int}} \; : \; t_{\mathrm{end}}$$

so that degrees of the vertices of $V(G)$ are split proportionally. This will allow us to embed the vertices of $W_A \cup W_B$ in $\mathfrak{P}_0$, the internal shrubs in $\mathfrak{P}_1$, and end shrubs in $\mathfrak{P}_2$. Actually, as our embedding procedure is more complex, we not only require the degrees to be split proportionally, but also to partition proportionally the objects from the bounded decomposition. In Section 7.5 we give some reasons why such a random splitting needs to be used.

Lemma 7.3 below is formulated in an abstract setting, without any reference to the tree T, and with a general number of classes in the partition.

Lemma 7.3. *For each $p \in \mathbb{N}$ and $a > 0$ there exists $k_0 > 0$ such that for each $k > k_0$ we have the following.*

Suppose G is a graph of order $n \geqslant k_0$ and $\deg^{\max}(G) \leqslant \Omega^* k$ with its $(k, \Lambda, \gamma, \varepsilon, k^{-0.05}, \rho)$-bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$. As usual, we write G_{∇} for the subgraph captured by $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$, and $G_{\mathcal{D}}$ for the spanning subgraph of G consisting of the edges in $\mathcal{D}$. Let $\mathcal{M}$ be an $(\varepsilon, d, k^{0.95})$-semiregular matching in G, and $\mathfrak{U}_1, \ldots, \mathfrak{U}_p$ be subsets of $V(G)$. Suppose that $\Omega^* \geqslant 1$ and $\Omega^*/\gamma < k^{0.1}$.

Suppose that $\mathfrak{q}_1, \ldots, \mathfrak{q}_p \in \{0\} \cup [a, 1]$ are reals with $\sum \mathfrak{q}_i \leqslant 1$. Then there exists a partition $\mathfrak{Q}_1 \cup \ldots \cup \mathfrak{Q}_p = V(G)$, and sets $\bar{V} \subseteq V(G)$, $\bar{\mathcal{V}} \subseteq \mathcal{V}(\mathcal{M})$, $\bar{\mathbf{V}} \subseteq \mathbf{V}$ with the following properties.

(1) $|\bar{V}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathcal{V}}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathbf{V}}| < \exp(-k^{0.1})n$.

(2) For each $i \in [p]$ and each $C \in \mathbf{V} \setminus \bar{\mathbf{V}}$ we have $|C \cap \mathfrak{Q}_i| \geqslant \mathfrak{q}_i |\mathfrak{Q}_i| - k^{0.9}$.

[14]Note that in general we apply a *sparse* decomposition (as opposed to a *bounded* decomposition) on the graph $G = G_{\triangleright T1.3}$, cf. Lemma 4.14. However, it turns out that when the vertices $\mathbf{\Psi}$ of huge degrees form a substantial part of G (which is when the need of transition from bounded to sparse decomposition arises), the result of this section is not needed.

(3) For each $i \in [p]$ and each $C \in \mathcal{V}(\mathcal{M}) \setminus \bar{\mathcal{V}}$ we have $|C \cap \mathfrak{Q}_i| \geqslant \mathfrak{q}_i |\mathfrak{Q}_i| - k^{0.9}$.

(4) For each $i \in [p]$, $D = (U, W; F) \in \mathcal{D}$ and $\mathrm{deg^{min}}_D(U \setminus \bar{V}, W \cap \mathfrak{Q}_i) \geqslant \mathfrak{q}_i \gamma k - k^{0.9}$.

(5) For each $i, j \in [p]$ we have $|\mathfrak{Q}_i \cap \mathfrak{U}_j| \geqslant \mathfrak{q}_i |\mathfrak{U}_j| - n^{0.9}$.

(6) For each $i \in [p]$ each $J \subseteq [p]$ and each $v \in V(G) \setminus \bar{V}$ we have

$$\deg_H(v, \mathfrak{Q}_i \cap \mathfrak{U}_J) \geqslant \mathfrak{q}_i \deg_H(v, \mathfrak{U}_J) - 2^{-p} k^{0.9} ,$$

for each of the graphs $H \in \{G, G_\nabla, G_{\exp}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$, where $\mathfrak{U}_J := \left(\bigcap_{j \in J} \mathfrak{U}_j\right) \setminus \left(\bigcup_{j \in [p] \setminus J} \mathfrak{U}_j\right)$.

(7) For each $i, i', j, j' \in [p]$ $(j \neq j')$, we have

$$\begin{aligned} e_H(\mathfrak{Q}_i \cap \mathfrak{U}_j, \mathfrak{Q}_{i'} \cap \mathfrak{U}_{j'}) &\geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e_H(\mathfrak{U}_j, \mathfrak{U}_{j'}) - k^{0.6} n^{0.6} , \\ e_H(\mathfrak{Q}_i \cap \mathfrak{U}_j, \mathfrak{Q}_{i'} \cap \mathfrak{U}_j) &\geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e(H[\mathfrak{U}_j]) - k^{0.6} n^{0.6} \qquad \text{if } i \neq i', \text{ and} \\ e(H[\mathfrak{Q}_i \cap \mathfrak{U}_j]) &\geqslant \mathfrak{q}_i^2 e(H[\mathfrak{U}_j]) - k^{0.6} n^{0.6} . \end{aligned}$$

for each of the graphs $H \in \{G, G_\nabla, G_{\exp}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$.

(8) For each $i \in [p]$ if $\mathfrak{q}_i = 0$ then $\mathfrak{Q}_i = \emptyset$.

Proof. We can assume that $\sum \mathfrak{q}_i = 1$ as all bounds in (2)–(7) are lower bounds. Assume that k is large enough. We assign each vertex $v \in V(G)$ to one of the sets $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_p$ at random with respective probabilities $\mathfrak{q}_1, \ldots, \mathfrak{q}_p$. Let $\bar{V}_1$ and $\bar{V}_2$ be the vertices which do not satisfy (4) and (6), respectively. Let $\bar{\mathcal{V}}$ be the sets of $\mathcal{V}(\mathcal{M})$ which do not satisfy (3), and let $\bar{\mathbf{V}}$ be the clusters of $\mathbf{V}$ which do not satisfy (2). Setting $\bar{V} := \bar{V}_1 \cup \bar{V}_2$, we need to show that (1), (5) and (7) are fulfilled simultaneously with positive probability. Using the union bound, it suffices to show that each of the properties (1), (5) and (7) is violated with probability at most 0.2. The probability of each of these three properties can be controlled in a straightforward way by the Chernoff bound. We only give such a bound (with error probability at most 0.1) on the size of the set $\bar{V}_1$ (appearing in (1)), which is the most difficult one to control.

For $i \in [p]$, let $\bar{V}_{1,i}$ be the set of vertices v for which there exists $D = (U, W; F) \in \mathcal{D}$, $U \ni v$, such that $\deg_D(v, W \cap \mathfrak{Q}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. We aim to show that for each $i \in [p]$ the probability that $|\bar{V}_{1,i}| > \exp(-k^{0.2})n$ is at most $\frac{1}{10p}$. Indeed, summing such an error bound together with similar bounds for other properties will allow us to conclude the statement. This will in turn follow from the Markov Inequality provided that we show that

$$\mathbf{E}[|\bar{V}_{1,i}|] \leqslant \frac{1}{10p} \cdot \exp(-k^{0.2})n . \tag{7.1}$$

Indeed, let us consider an arbitrary vertex $v \in V(G)$. By Fact 4.3, v is contained in at most Ω^*/γ dense spots of $\mathcal{D}$. For a fixed dense spot $D = (U, W; F) \in \mathcal{D}$ with $v \in U$ let us bound the probability of the event $\mathcal{E}_{v,i,D}$ that $\deg_D(v, W \cap \mathfrak{Q}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. To this end, fix a set $N \subseteq W \cap \mathrm{N}_D(v)$ of size exactly γk before the random assignment is performed. Now, elements of $V(G)$ are distributed

randomly into the sets $\mathfrak{Q}_1,\ldots,\mathfrak{Q}_p$. In particular, the number $|\mathfrak{Q}_i\cap N|$ has binomial distribution with parameters γk and $\mathfrak{q}_i$. Using the Chernoff bound, we get

$$\mathbf{P}[\mathcal{E}_{v,i,D}]\leqslant\mathbf{P}\left[|\mathfrak{Q}_i\cap N|<\mathfrak{q}_i\gamma k-k^{0.9}\right]\leqslant\exp(-k^{0.3})\;.$$

Thus, it follows by summing the tail over at most $\Omega^*/\gamma\leqslant k^{0.1}$ dense spots containing v, that

$$\mathbf{P}[v\in\bar{V}_{1,i}]\leqslant k^{0.1}\cdot\exp(-k^{0.3})\;. \tag{7.2}$$

Now, (7.1) follows by linearity of expectation. □

Lemma 7.3 is utilized for the purpose of our proof of Theorem 1.3 using the notion of proportional partition introduced in Definition 7.6 below.

7.3 Common settings

Throughout Section 7 and Section 8 we shall be working with the setting that comes from Lemma 6.1. In order to keep statements of the subsequent lemmas reasonably short we introduce the following setting.

Setting 7.4. *We assume that the constants $\Lambda,\Omega^*,\Omega^{**},k_0$ and $\widehat{\alpha},\gamma,\varepsilon,\varepsilon',\eta,\pi,\rho,\tau,d$ satisfy*

$$\eta\gg\frac{1}{\Omega^*}\gg\frac{1}{\Omega^{**}}\gg\rho\gg\gamma\gg d\geqslant\frac{1}{\Lambda}\geqslant\varepsilon\geqslant\pi\geqslant\widehat{\alpha}\geqslant\varepsilon'\geqslant\nu\gg\tau\gg\frac{1}{k_0}>0\;, \tag{7.3}$$

and that $k\geqslant k_0$. Here, by writing $c>a_1\gg a_2\gg\ldots\gg a_\ell>0$ we mean that there exist non-decreasing functions $f_i:(0,c)^i\to(0,c)$ ($i=1,\ldots,\ell-1$) such that for each $i\in[\ell-1]$ we have $a_{i+1}<f_i(a_1,\ldots,a_i)$. [15]

Suppose that $G\in\mathbf{LKSsmall}(n,k,\eta)$ is given together with its $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon',\nu,\rho)$-sparse decomposition

$$\nabla=(\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})\;,$$

with respect to the partition $\{\mathbb{S}_{\eta,k}(G),\mathbb{L}_{\eta,k}(G)\}$, and with respect to the avoiding threshold $\frac{\rho k}{100\Omega^*}$. We write

$$V_{\leadsto\mathfrak{A}}:=\mathbf{shadow}_{G_\nabla-\mathbf{\Psi}}(\mathfrak{A},\frac{\rho k}{100\Omega^*})\quad\text{and}\quad\mathbf{V}_{\leadsto\mathfrak{A}}:=\{C\in\mathbf{V}\;:\;C\subseteq V_{\leadsto\mathfrak{A}}\}\;. \tag{7.4}$$

The graph $\mathbf{G}_{\mathrm{reg}}$ is the corresponding cluster graph. Let $\mathfrak{c}$ be the size of an arbitrary cluster in $\mathbf{V}$.[16] *Let G_∇ be the spanning subgraph of G formed by the edges captured by ∇. There are two $(\varepsilon,d,\pi\mathfrak{c})$-semiregular matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ in $G_{\mathcal{D}}$, with the following properties (we abbreviate $\mathbb{XA}:=\mathbb{XA}(\eta,\nabla,\mathcal{M}_A,\mathcal{M}_B)$, $\mathbb{XB}:=\mathbb{XB}(\eta,\nabla,\mathcal{M}_A,\mathcal{M}_B)$, and $\mathbb{XC}:=\mathbb{XC}(\eta,\nabla,\mathcal{M}_A,\mathcal{M}_B)$):*

1. *$V(\mathcal{M}_A)\cap V(\mathcal{M}_B)=\emptyset$,*

[15] The precise relation between the parameters can be found on page 154, with $\Omega^{**}:=\Omega_{j+1}$ and $\Omega^*:=\Omega_j$ for a certain index $j\in[g]$ to be specified in the course of the proof there.

[16] The number $\mathfrak{c}$ is not defined when $\mathbf{V}=\emptyset$. However in that case $\mathfrak{c}$ is never actually used.

2. $V_1(\mathcal{M}_B) \subseteq S^0$, *where*
$$S^0 := \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A}) , \tag{7.5}$$

3. *for each* $(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B$, *there is a dense spot* $(U, W; F) \in \mathcal{D}$ *with* $X \subseteq U$ *and* $Y \subseteq W$, *and further, either* $X \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $X \subseteq \mathbb{L}_{\eta,k}(G)$, *and* $Y \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $Y \subseteq \mathbb{L}_{\eta,k}(G)$,

4. *for each* $X_1 \in \mathcal{V}_1(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists a cluster* $C_1 \in \mathbf{V}$ *such that* $X_1 \subseteq C_1$, *and for each* $X_2 \in \mathcal{V}_2(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists* $C_2 \in \mathbf{V} \cup \{\mathbb{L}_{\eta,k}(G) \cap \mathfrak{A}\}$ *such that* $X_2 \subseteq C_2$,

5. *each pair of the semiregular matching* $\mathcal{M}_{\mathrm{good}} := \{(X_1, X_2) \in \mathcal{M}_A \ : \ X_1 \cup X_2 \subseteq \mathbb{XA}\}$ *corresponds to an edge in* $\mathbf{G}_{\mathrm{reg}}$,

6. $e_{G_\nabla}\big(\mathbb{XA}, S^0 \setminus V(\mathcal{M}_A)\big) \leqslant \gamma k n$,

7. $e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \gamma^2 k n$,

8. *for the semiregular matching* $\mathcal{N}_{\mathfrak{A}} := \{(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B \ : \ (X \cup Y) \cap \mathfrak{A} \neq \emptyset\}$ *we have* $e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})\big) \leqslant \gamma^2 k n$,

9. $|E(G) \setminus E(G_\nabla)| \leqslant 2\rho k n$,

10. $|E(G_{\mathcal{D}}) \setminus (E(G_{\mathrm{reg}}) \cup E_G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])| \leqslant \frac{5}{4}\gamma k n$.

We write

$$V_+ := V(G) \setminus (S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \tag{7.6}$$
$$= \mathbb{L}_{\eta,k}(G) \cup V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B) , \tag{7.7}$$
$$L_\# := \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \text{ , and} \tag{7.8}$$
$$V_{\mathrm{good}} := V_+ \setminus (\mathbf{\Psi} \cup L_\#) , \tag{7.9}$$
$$\mathbb{YA} := \mathbf{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, (1 + \frac{\eta}{10})k\right) \setminus \mathbf{shadow}_{G - G_\nabla}\left(V(G), \frac{\eta}{100}k\right) , \tag{7.10}$$
$$\mathbb{YB} := \mathbf{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, (1 + \frac{\eta}{10})\frac{k}{2}\right) \setminus \mathbf{shadow}_{G - G_\nabla}\left(V(G), \frac{\eta}{100}k\right) , \tag{7.11}$$
$$V_{\not\leadsto\mathbf{\Psi}} := (\mathbb{XA} \cup \mathbb{XB}) \cap \mathbf{shadow}_G\left(\mathbf{\Psi}, \frac{\eta}{100}k\right) , \tag{7.12}$$
$$\mathtt{P}_{\mathfrak{A}} := \mathbf{shadow}_{G_{\mathrm{reg}}}(V(\mathcal{N}_{\mathfrak{A}}), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B) ,$$
$$\mathtt{P}_1 := \mathbf{shadow}_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B) ,$$
$$\mathtt{P} := (\mathbb{XA} \setminus \mathbb{YA}) \cup ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_1$$
$$\cup\, \mathbf{shadow}_{G_{\mathcal{D}} \cup G_\nabla}(V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_{\mathfrak{A}} \cup \mathtt{P}_1, \frac{\eta^2 k}{10^5}) ,$$
$$\mathtt{P}_2 := \mathbb{XA} \cap \mathbf{shadow}_{G_\nabla}(S^0 \setminus V(\mathcal{M}_A), \sqrt{\gamma}k) ,$$
$$\mathtt{P}_3 := \mathbb{XA} \cap \mathbf{shadow}_{G_\nabla}(\mathbb{XA}, \eta^3 k/10^3) ,$$
$$\mathcal{F} := \{C \in \mathcal{V}(\mathcal{M}_A) : C \subseteq \mathbb{XA}\} \cup \mathcal{V}_1(\mathcal{M}_B) . \tag{7.13}$$

The vertex set $\mathbb{YA}$ in Setting 7.4 should be regarded as $\mathbb{XA}$ cleaned from rare irregularities. Indeed, as it turns out most of the vertices from $\mathbb{XA}$ are contained in $\mathbb{YA}$. Likewise, $\mathbb{YB}$ should be regarded as a cleaned version of $\mathbb{XA} \cup \mathbb{XB}$. These properties are stated in Lemma 7.9 below.

On the interface between Lemma 7.32 and Lemma 7.35 we shall need to work with a semiregular matching which is formed of only those edges $E(\mathcal{D})$ which are either incident with $\mathfrak{A}$, or included in G_{reg}. The following lemma provides us with an appropriate "cleaned version of $\mathcal{D}$". The notion of being absorbed adapts in a straightforward way to two families of dense spots: a family of dense spots $\mathcal{D}_1$ *is absorbed* by another family $\mathcal{D}_2$ if for every $D_1 \in \mathcal{D}_2$ there exists $D_2 \in \mathcal{D}_2$ such that D_1 is contained in D_2 as a subgraph.

Lemma 7.5. *Assume Setting 7.4. Then there exists a family $\mathcal{D}_\nabla$ of edge-disjoint $(\gamma^3 k/4, \gamma/2)$-dense spots absorbed by $\mathcal{D}$ such that*

1. *$|E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| \leqslant \rho k n$, and*
2. *$E(\mathcal{D}_\nabla) \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])$.*

The proof of Lemma 7.5 is a warm-up for proofs in Section 7.6.

Proof of Lemma 7.5. We discard those dense spots $D \in \mathcal{D}$ for which

$$\left|E(D) \setminus (E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])\right| \geqslant \sqrt{\gamma} e(D) \ . \tag{7.14}$$

For each remaining dense spot $D \in \mathcal{D}$ we show below how to extract a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$ with $e(D') \geqslant (1 - 2\sqrt{\gamma}) e(D)$ and $E(D) \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])$. Let $\mathcal{D}_\nabla$ be the set of all thus obtained D'. This way we ensure Property 2, and we also have Property 1, since

$$\begin{aligned} |E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| &\leqslant \frac{1}{\sqrt{\gamma}} \left| E(\mathcal{D}) \setminus \left(E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])\right)\right| + 2\sqrt{\gamma} \cdot e(\mathcal{D}) \\ {\scriptstyle (\text{by S7.4(10), and as } e(\mathcal{D}) \leqslant e(G) \leqslant kn)} \quad &\leqslant 3\rho k n \ . \end{aligned}$$

We now show how to extract a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$ with $e(D') \geqslant (1 - 2\sqrt{\gamma}) e(D)$ and $E(D) \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])$ from any spot $D \in \mathcal{D}$ which does not satisfy (7.14). Let $D = (A, B; F)$, and $a := |A|$, $b := |B|$. As D is $(\gamma k, \gamma)$-dense, we have $a, b \geqslant \gamma k$. First, we discard from D all edges not contained in $E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])$ to obtain a dense spot $D^* \subseteq D$ with $e(D^*) \geqslant (1 - \sqrt{\gamma}) e(D)$. Next, we perform a sequential cleaning procedure in D^*. As long as there are such vertices, discard from A any vertex whose current degree is less than $\gamma^2 b/4$, and discard from B any vertex whose current degree is less than $\gamma^2 a/4$. When this procedure terminates, the resulting graph $D' = (A', B'; F')$ has ${\deg^{\min}}_{D'}(A') \geqslant \gamma^2 b/4 \geqslant \gamma^3 k/4$ and ${\deg^{\min}}_{D'}(B') \geqslant \gamma^3 k/4$. Note that we deleted at most $a \cdot \gamma^2 b/4 + b \cdot \gamma^2 a/4$ edges out of the at least $(1 - \sqrt{\gamma}) e(D)$ edges of D^*. This means that $e(D') \geqslant (1 - \sqrt{\gamma}) e(D) - \gamma^2 ab/2 \geqslant (1 - 2\sqrt{\gamma}) e(D)$, as desired. Thus we also have the required density of D', namely $\mathrm{d}_{D'}(A', B') \geqslant (1 - 2\sqrt{\gamma})\gamma \geqslant \gamma/2$. $\square$

In some cases, we shall in addition partition the set $V(G)$ into three sets as in Lemma 7.3. This motivates the following definition.

Definition 7.6 (Proportional splitting). *Let $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 > 0$ be three positive reals with $\sum_i \mathfrak{p}_i \leqslant 1$. Under Setting 7.4, suppose that $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ is a partition of $V(G) \setminus \mathbf{\Psi}$ which satisfies assertions of Lemma 7.3 with parameter $p_{\triangleright \mathrm{L}7.3} := 10$ for graph $G^*_{\triangleright \mathrm{L}7.3} := (G_\nabla - \mathbf{\Psi}) \cup G_\mathcal{D}$ (here, by the union, we mean union of the edges), bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$, matching $\mathcal{M}_{\triangleright \mathrm{L}7.3} := \mathcal{M}_A \cup \mathcal{M}_B$, sets $\mathfrak{U}_1 := V_{\mathrm{good}}, \mathfrak{U}_2 := \mathbb{XA} \setminus (\mathbf{\Psi} \cup \mathtt{P})$, $\mathfrak{U}_3 := \mathbb{XB} \setminus \mathtt{P}$, $\mathfrak{U}_4 := V(G_{\mathrm{exp}})$, $\mathfrak{U}_5 := \mathfrak{A}$, $\mathfrak{U}_6 := V_{\leadsto \mathfrak{A}}$, $\mathfrak{U}_7 := \mathtt{P}_{\mathfrak{A}}$, $\mathfrak{U}_8 := \mathbb{L}_{\eta,k}(G)$, $\mathfrak{U}_9 := L_\#$, $\mathfrak{U}_{10} := V_{\not\leadsto \mathbf{\Psi}}$ and reals $\mathfrak{q}_1 := \mathfrak{p}_0, \mathfrak{q}_2 := \mathfrak{p}_1$, $\mathfrak{q}_3 := \mathfrak{p}_2$, $\mathfrak{q}_4 := \ldots \mathfrak{q}_{10} = 0$. Note that by Lemma 7.3(8) we have that $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ is a partition of $V(G) \setminus \mathbf{\Psi}$. We call $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$* proportional *$(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$* splitting.

We refer to properties of the proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ using the numbering of Lemma 7.3; for example, "Definition 7.6(5)" tells us among other things that $|(\mathbb{XA} \setminus \mathtt{P}) \cap \mathfrak{P}_0| \geqslant \mathfrak{p}_0 |\mathbb{XA} \setminus (\mathtt{P} \cup \mathbf{\Psi})| - n^{0.9}$.

Setting 7.7. *Under Setting 7.4, suppose that we are given a proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ of $V(G) \setminus \mathbf{\Psi}$. We assume that*

$$\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 \geqslant \frac{\eta}{100} . \tag{7.15}$$

Let $\bar{V}, \bar{\mathcal{V}}, \bar{\mathbf{V}}$ be the exceptional sets as in Definition 7.6(1).

We write

$$\mathbb{F} := \mathbf{shadow}_{G_\mathcal{D}} \left(\bigcup \bar{\mathcal{V}} \cup \bigcup \bar{\mathcal{V}}^* \cup \bigcup \bar{\mathbf{V}}, \frac{\eta^2 k}{10^{10}} \right) , \tag{7.16}$$

where $\bar{\mathcal{V}}^*$ are the partners of $\bar{\mathcal{V}}$ in $\mathcal{M}_A \cup \mathcal{M}_B$.

We have

$$|\mathbb{F}| \leqslant \varepsilon n . \tag{7.17}$$

For an arbitrary set $U \subseteq V(G)$ and for $i \in \{0,1,2\}$ we write $U^{\restriction i}$ for the set $U \cap \mathfrak{P}_i$.

For each $(X,Y) \in \mathcal{M}_A \cup \mathcal{M}_B$ such that $X, Y \notin \bar{\mathcal{V}}$ we write $(X,Y)^{\restriction i}$ for an arbitrary fixed pair $(X' \subseteq X, Y' \subseteq Y)$ with the property that $|X'| = |Y'| = \min\{|X^{\restriction i}|, |Y^{\restriction i}|\}$. We extend this notion of restriction to an arbitrary semiregular matching $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$ as follows. We set

$$\mathcal{N}^{\restriction i} := \left\{ (X,Y)^{\restriction i} \,:\, (X,Y) \in \mathcal{N} \text{ with } X, Y \notin \bar{\mathcal{V}} \right\} .$$

The next lemma provides some simple properties of a restriction of a semiregular matching.

Lemma 7.8. *Assume Setting 7.7. Then for each $i \in \{0,1,2\}$, and for each $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$ we have that $\mathcal{N}^{\restriction i}$ is a $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi}{200}\mathfrak{c})$-semiregular matching satisfying*

$$|V(\mathcal{N}^{\restriction i})| \geqslant \mathfrak{p}_i |V(\mathcal{N})| - 2k^{-0.05} n . \tag{7.18}$$

Moreover for all $v \notin \mathbb{F}$ and for all $i = 0,1,2$ we have $\deg_{G_\mathcal{D}}(v, V(\mathcal{N})^{\restriction i} \setminus V(\mathcal{N}^{\restriction i})) \leqslant \frac{\eta^2 k}{10^5}$.

Proof. Let us consider an arbitrary pair $(X,Y) \in \mathcal{N}$. By Definition 7.6(3) we have

$$|X^{\restriction i}| \geqslant \mathfrak{p}_i |X| - k^{0.9} \overset{(7.15)}{\geqslant} \frac{\eta}{200}|X| \quad \text{and} \quad |Y^{\restriction i}| \geqslant \mathfrak{p}_i |Y| - k^{0.9} \overset{(7.15)}{\geqslant} \frac{\eta}{200}|Y| . \tag{7.19}$$

In particular, Fact 2.7 gives that $(X,Y)^{\restriction i}$ is a $400\varepsilon/\eta$-regular pair of density at least $d/2$.

We now turn to (7.18). The total order of pairs $(X,Y) \in \mathcal{N}$ excluded entirely from $\mathcal{N}^{\restriction 1}$ is at most $2\exp(-k^{0.1})n < k^{-0.05}n$ by Definition 7.6(1). Further, for each $(X,Y) \in \mathcal{N}$ whose part is included to $\mathcal{N}^{\restriction 1}$ we have by that $|V((X,Y)^{\restriction i})| \geqslant \mathfrak{p}_i(|X|+|Y|) - 2k^{0.9}$ by (7.19). As $|\mathcal{N}| \leqslant \frac{n}{2k^{0.95}}$, and (7.18) follows.

For the moreover part, note that by Fact 4.3 and Fact 4.4

$$\deg_{G_{\mathcal{D}}}(v, V(\mathcal{N})^{\restriction i} \setminus V(\mathcal{N}^{\restriction i})) \leqslant \frac{\eta^2 k}{10^{10}} + \frac{(\Omega^*)^2}{\pi\nu\gamma^2} \cdot 3k^{0.9} \leqslant \frac{\eta^2 k}{10^5} .$$

□

The following lemma gives a useful bound on some of the sets defined on page 68.

Lemma 7.9. *Suppose we are in Setting 7.4. Suppose that all but at most βkn edges are captured by ∇. Then,*

$$|L_{\#}| \leqslant \frac{20\beta}{\eta} n \tag{7.20}$$

$$|\mathbb{XA} \setminus \mathbb{YA}| \leqslant \frac{600\beta}{\eta^2} n \text{ , and} \tag{7.21}$$

$$|(\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}| \leqslant \frac{600\beta}{\eta^2} n \ . \tag{7.22}$$

Further, if $e_G(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \leqslant \tilde{\beta}kn$ then

$$|V_{\not\to\mathbf{\Psi}}| \leqslant \frac{100\tilde{\beta}n}{\eta} \ . \tag{7.23}$$

Proof. Let $W_1 := \{v \in V(G) : \deg_G(v) - \deg_{G_\nabla}(v) \geqslant \eta k/100\}$. We have $|W_1| \leqslant \frac{200\beta}{\eta} n$.

Observe that $L_{\#}$ sends out at most $(1+\frac{9}{10}\eta)k|L_{\#}| < \frac{40\beta}{\eta}kn$ edges in G_∇. Let $W_2 := \{v \in V(G) : \deg_{G_\nabla}(v, L_{\#}) \geqslant \eta k/10\}$. We have $|W_2| \leqslant \frac{400\beta}{\eta^2} n$.

Let $W_3 := \{v \in \mathbb{XA} : \deg_{G_\nabla}(v, S^0 \setminus V(\mathcal{M}_A)) \geqslant \sqrt{\gamma}k\}$. By Property 6 we have $|W_3| \leqslant \sqrt{\gamma}n$.

Now, observe that $\mathbb{XA} \setminus \mathbb{YA} \subseteq W_1 \cup W_2 \cup W_3$, and $\mathbb{XB} \setminus \mathbb{YB} \subseteq W_1 \cup W_2$.

The bound (7.23) follows in a straightforward way. □

We finish this section with an auxiliary result which will only be used later in the proofs of Lemmas 7.34 and 7.35.

Lemma 7.10. *Assume Settings 7.4 and 7.7. We have that for $i = 1, 2$*

$$\mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \mathbb{F}) \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_{\mathcal{D}}}(V_{\not\to\mathbf{\Psi}}, \frac{\eta^2 k}{10^5})\right) , \tag{7.24}$$

$$\deg^{\min}{}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V}), V_{\text{good}}^{\restriction i}\right) \geqslant \mathfrak{p}_i(1+\frac{\eta}{20})k , \tag{7.25}$$

$$\deg^{\min}{}_{G_\nabla}\left(\mathbb{XB} \setminus (\mathtt{P} \cup \bar{V}), V_{\text{good}}^{\restriction i}\right) \geqslant \mathfrak{p}_i(1+\frac{\eta}{20})\frac{k}{2} \text{ , and} \tag{7.26}$$

$$\deg^{\max}{}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathtt{P}_2 \cup \mathtt{P}_3), \bigcup \mathcal{F}\right) \leqslant \frac{3\eta^3}{2 \cdot 10^3} k \ . \tag{7.27}$$

Moreover, $\mathcal{F}$ defined in (7.13) *is an* $(\mathcal{M}_A \cup \mathcal{M}_B)$*-cover.*

Proof. The definition of $\mathtt{P}$ gives (7.24).

For (7.25) and (7.26), assume that $i = 2$ (the other case is analogous). Observe that

$$\begin{aligned}
&\deg^{\min}{}_{G_\nabla}\left(\mathbb{YA} \setminus (V_{\not\leadsto\mathbf{\Psi}} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \\
\text{(by Def 7.6(6))} \quad &\geqslant \mathfrak{p}_2 \cdot \deg^{\min}{}_{G_\nabla}(\mathbb{YA} \setminus V_{\not\leadsto\mathbf{\Psi}}, V_{\mathrm{good}}) - k^{0.9} \\
\text{(by (7.9))} \quad &\geqslant \mathfrak{p}_2 \cdot \left(\deg^{\min}{}_{G_\nabla}(\mathbb{YA}, V_+ \setminus L_\#) - \deg^{\max}{}_{G_\nabla}(\mathbb{YA} \setminus V_{\not\leadsto\mathbf{\Psi}}, \mathbf{\Psi})\right) - k^{0.9} \\
\text{(by (7.10), (7.12))} \quad &\geqslant \mathfrak{p}_2 \cdot \left((1 + \frac{\eta}{10})k - \frac{\eta k}{100}\right) - k^{0.9} \\
\text{(by (7.3), (7.15))} \quad &\geqslant \mathfrak{p}_2 \cdot (1 + \frac{\eta}{20})k \ ,
\end{aligned}$$

which proves (7.25), as $\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V}) \subseteq \mathbb{YA} \setminus (V_{\not\leadsto\mathbf{\Psi}} \cup \bar{V})$. Similarly, we obtain that

$$\deg^{\min}{}_{G_\nabla}\left(\mathbb{YB} \setminus (V_{\not\leadsto\mathbf{\Psi}} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \geqslant \mathfrak{p}_2(1 + \frac{\eta}{20})\frac{k}{2} \ ,$$

which proves (7.26).

We have $\deg^{\max}{}_{G_\nabla}(\mathbb{XA} \setminus \mathtt{P}_3, \mathbb{XA}) < \frac{\eta^3}{10^3}k$, and $\deg^{\max}{}_{G_\nabla}(\mathbb{XA} \setminus \mathtt{P}_2, S^0 \setminus V(\mathcal{M}_A)) < \sqrt{\gamma}k$. Thus (7.27) follows from Setting 7.4(2) and by (7.3).

For the "moreover" part, it suffices to prove that $\{C \in \mathcal{V}(\mathcal{M}_A) : C \subseteq \mathbb{XA}\} = \mathcal{F} \setminus \mathcal{V}_1(\mathcal{M}_B)$ is an $\mathcal{M}_A$-cover. Let $(T_1, T_2) \subseteq \mathcal{M}_A$. As $G \in \mathbf{LKSsmall}(n, k, \eta)$, we have by Setting 7.4(3) that for some $i \in \{1, 2\}$, T_i is contained in $\mathbb{L}_{\eta,k}(G)$. Then by Setting 7.4(1), $T_i \subseteq \mathbb{XA}$, as desired. □

7.4 Types of configurations

We can now define the following preconfigurations (♣), (♡**1**), (♡**2**), (**exp**), and (**reg**), and the configurations[17] (⋄**1**)–(⋄**10**). It will follow from results from other sections that at least one of the configurations (⋄**1**)–(⋄**10**) appears in each graph $\mathbf{LKS}(n, k, \eta)$. More precisely, after getting the "rough structure" in Lemma 6.1 we get one of the configurations (⋄**1**)–(⋄**10**) from Lemma 7.32. The latter lemma reduces the situation to one of three cases which are then dealt with in Lemmas 7.33–7.35 separately. Then, in Section 8, we provide with an embedding for a given tree $T_{\triangleright \mathrm{T}1.3} \in \mathbf{trees}(k)$.

We now give a brief overview of these configurations. Configuration (⋄**1**) covers the easy and lucky case when G is contains a subgraph with high minimum degree. A very simple tree-embedding strategy similar to the greedy strategy turns out to work in this case.

The purpose of Preconfiguration (♣) is to utilize vertices of $\mathbf{\Psi}$. On one hand these vertices seem very powerful because of their large degree, on the other hand the edges incident with them are very unstructured. Therefore Preconfiguration (♣) distills some structure in $\mathbf{\Psi}$. This preconfiguration is then a part of configurations (⋄**2**)–(⋄**5**) which deal with the case when $\mathbf{\Psi}$ is substantial. Indeed,

[17]The word "configuration" is used for a final structure in a graph which is suitable for embedding purposes while "preconfigurations" are building blocks for configurations.

Lemma 7.33 asserts that whenever $\mathbf{\Psi}$ is incident with many edges in the setting provided by Lemma 6.1, at least one of configurations ($\diamond\mathbf{1}$)–($\diamond\mathbf{5}$) must occur.

The cases when the number of edges incident with $\mathbf{\Psi}$ is negligible are covered by configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{10}$). More precisely, in this setting Lemma 7.32 transforms the output structure of Lemma 6.1 into an input structure for either Lemma 7.34 or Lemma 7.35. These lemmas then assert that indeed one of the Configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{10}$) must occur. The configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{8}$) involve combinations of one of the two preconfigurations ($\heartsuit\mathbf{1}$) and ($\heartsuit\mathbf{2}$) and one of the two preconfigurations (**exp**) and (**reg**). The idea here is that the knags are embedded using the structure of (**exp**) or (**reg**) (whichever applicable), the internal shrubs are embedded using the structure which is specific to each of the configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{8}$), and the end shrubs are embedded using the structure of ($\heartsuit\mathbf{1}$) or ($\heartsuit\mathbf{2}$). The configuration ($\diamond\mathbf{10}$) is very similar to the structures obtained in the dense setting in [PS12, HP] (see Section 8.1.5 for a discussion), and ($\diamond\mathbf{9}$) should be considered as half-way towards it.

The reader may find it helpful to compare the definitions of the configurations with Section 8.1 where an overview is given how these configurations are used to embed the tree $T_{\triangleright\mathrm{T}1.3}$.

Some of the configurations below are accompanied with parameters in the parentheses; note that we do not make explicit those numerical parameters which are inherited from Setting 7.4.

We start off by giving definitions of Configuration ($\diamond\mathbf{1}$). This is a very easy configuration in which a modification of the greedy tree-embedding strategy works.

Definition 7.11 (**Configuration** ($\diamond$**1**))**.** *We say that a graph G is in* Configuration ($\diamond\mathbf{1}$) *if there exists a non-empty bipartite graph $H \subseteq G$ with* $\mathrm{deg^{min}}_G(V(H)) \geqslant k$ *and* $\mathrm{deg^{min}}(H) \geqslant k/2$.

We now introduce the configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$) which make use of the set $\mathbf{\Psi}$. These configurations build on Preconfiguration ($\clubsuit$). Figure 8.1 shows common features of the configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$).

Definition 7.12 (**Preconfiguration** ($\clubsuit$))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Preconfiguration ($\clubsuit$)($\Omega^\star$) *if the following conditions are met. G contains non-empty sets $L'' \subseteq L' \subseteq \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus \mathbf{\Psi}$, and a non-empty set $\mathbf{\Psi}' \subseteq \mathbf{\Psi}$ such that*

$$\mathrm{deg^{max}}_{G_\nabla}(L', \mathbf{\Psi} \setminus \mathbf{\Psi}') < \frac{\eta k}{100}\ , \tag{7.28}$$

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}', L'') \geqslant \Omega^\star k\ ,\ \textit{and} \tag{7.29}$$

$$\mathrm{deg^{max}}_{G_\nabla}(L'', \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus (\mathbf{\Psi} \cup L')) \leqslant \frac{\eta k}{100}\ . \tag{7.30}$$

Definition 7.13 (**Configuration** ($\diamond$**2**))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration ($\diamond\mathbf{2}$)($\Omega^\star, \tilde{\Omega}, \beta$) *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, a set $V_1 \subseteq V(G_{\mathrm{exp}}) \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G_{\mathrm{exp}})$ with the following properties.

$$\begin{aligned}\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) &\geqslant \tilde{\Omega}k\\ \mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') &\geqslant \beta k\ ,\\ \mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, V_2) &\geqslant \beta k\ ,\\ \mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_2, V_1) &\geqslant \beta k\ .\end{aligned}$$

Definition 7.14 (Configuration $(\diamond\mathbf{3})$). *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration $(\diamond\mathbf{3})(\Omega^\star, \tilde{\Omega}, \zeta, \delta)$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exist a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, a set $V_1 \subseteq \mathfrak{A} \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G) \setminus \mathbf{\Psi}$ such that the following properties are satisfied.

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega}k\ ,$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k\ ,$$

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}(V_1, V(G) \setminus (V_2 \cup \mathbf{\Psi})) \leqslant \zeta k\ , \tag{7.31}$$

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_2, V_1) \geqslant \delta k\ . \tag{7.32}$$

Definition 7.15 (Configuration $(\diamond\mathbf{4})$). *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration $(\diamond\mathbf{4})(\Omega^\star, \tilde{\Omega}, \zeta, \delta)$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exists a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, sets $V_1 \subseteq \mathbb{YB} \cap L''$, $\mathfrak{A}' \subseteq \mathfrak{A}$, and $V_2 \subseteq V(G) \setminus \mathbf{\Psi}$ with the following properties

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega}k\ ,$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k\ ,$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(V_1, \mathfrak{A}') \geqslant \delta k\ , \tag{7.33}$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(\mathfrak{A}', V_1) \geqslant \delta k\ , \tag{7.34}$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(V_2, \mathfrak{A}') \geqslant \delta k\ , \tag{7.35}$$

$$\mathrm{deg^{max}}_{G_\nabla \cup G_{\mathcal{D}}}(\mathfrak{A}', V(G) \setminus (\mathbf{\Psi} \cup V_2)) \leqslant \zeta k\ . \tag{7.36}$$

Definition 7.16 (Configuration $(\diamond\mathbf{5})$). *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration $(\diamond\mathbf{5})(\Omega^\star, \tilde{\Omega}, \delta, \zeta, \tilde{\pi})$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration $(\clubsuit)(\Omega^\star)$ in G. There exists a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, and a set $V_1 \subseteq (\mathbb{YB} \cap L'' \cap \bigcup \mathbf{V}) \setminus V(G_{\mathrm{exp}})$ such that the following conditions are fulfilled.

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega}k\ , \tag{7.37}$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k\ , \tag{7.38}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{reg}}}(V_1) \geqslant \zeta k\ . \tag{7.39}$$

Further, we have

$$C \cap V_1 = \emptyset \text{ or } |C \cap V_1| \geqslant \tilde{\pi}|C| \tag{7.40}$$

for every $C \in \mathbf{V}$.

In remains to introduce configurations ($\diamond$**6**)–($\diamond$**10**). In these configurations the set $\mathbf{\Psi}$ is not utilized. All these configurations make use of Setting 7.7, i.e., the set $V(G) \setminus \mathbf{\Psi}$ is partitioned into three sets $\mathfrak{P}_0, \mathfrak{P}_1$ and $\mathfrak{P}_2$. The purpose of $\mathfrak{P}_0, \mathfrak{P}_1$ and $\mathfrak{P}_2$ is to make possible to embed the knags, the internal shrubs, and the end shrubs of $T_{\triangleright \mathrm{T1.3}}$, respectively. Thus the parameters $\mathfrak{p}_0, \mathfrak{p}_1$ and $\mathfrak{p}_2$ are chosen proportionally to the sizes of these respective parts of $T_{\triangleright \mathrm{T1.3}}$. A summary picture for Configurations ($\diamond$**6**)–($\diamond$**7**), ($\diamond$**8**), and ($\diamond$**9**) is given in Figures 8.2, 8.3 and 8.4, respectively.

We first introduce four preconfigurations ($\heartsuit$**1**), ($\heartsuit$**2**), (**exp**) and (**reg**) which are building bricks for configurations ($\diamond$**6**)–($\diamond$**9**). The preconfigurations ($\heartsuit$**1**) and ($\heartsuit$**2**) will be used for embedding end shrubs of a fine partition of the tree $T_{\triangleright \mathrm{T1.3}}$, and preconfigurations (**exp**) and (**reg**) will be used for embedding its knags.

An $\mathcal{M}$-*cover* of a semiregular matching $\mathcal{M}$ is a family $\mathcal{F} \subseteq \mathcal{V}(\mathcal{M})$ with the property that at least one of the elements S_1 and S_2 is a member of $\mathcal{F}$, for each $(S_1, S_2) \in \mathcal{M}$.

Definition 7.17 (**Preconfiguration** ($\heartsuit$**1**))**.** *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration ($\heartsuit$**1**)(γ', h) *of $V(G)$ if there are two non-empty sets $V_0, V_1 \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_{\mathcal{D}}}(V_{\not\leadsto\mathbf{\Psi}}, \frac{\eta^2 k}{10^5})\right)$ with the following properties.*

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_0, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h/2 \text{ , and} \tag{7.41}$$

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h \text{ .} \tag{7.42}$$

Further, there is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{F}$ such that

$$\mathrm{deg^{max}}_{G_\nabla}\left(V_1, \bigcup \mathcal{F}\right) \leqslant \gamma' k \text{ .} \tag{7.43}$$

Definition 7.18 (**Preconfiguration** ($\heartsuit$**2**))**.** *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration ($\heartsuit$**2**)(h) *of $V(G)$ if there are two non-empty sets $V_0, V_1 \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_{\mathcal{D}}}(V_{\not\leadsto\mathbf{\Psi}}, \frac{\eta^2 k}{10^5})\right)$ with the following properties.*

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_0 \cup V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h. \tag{7.44}$$

Definition 7.19 (**Preconfiguration** (**exp**))**.** *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration (**exp**)(β) *if there are two non-empty sets $V_0, V_1 \subseteq \mathfrak{P}_0$ with the following properties.*

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_0, V_1) \geqslant \beta k \text{ ,} \tag{7.45}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, V_0) \geqslant \beta k \text{ .} \tag{7.46}$$

Definition 7.20 (**Preconfiguration** (**reg**))**.** *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ *if there are two non-empty sets $V_0, V_1 \subseteq \mathfrak{P}_0$ and a non-empty family of vertex-disjoint $(\tilde{\varepsilon}, d')$-super-regular pairs $\{(Q_0^{(j)}, Q_1^{(j)}\}_{j \in \mathcal{Y}}$ (with respect to the edge set $E(G)$) with $V_0 := \bigcup Q_0^{(j)}$ and $V_1 := \bigcup Q_1^{(j)}$ such that*

$$\min \left\{ |Q_0^{(j)}|, |Q_1^{(j)}| \right\} \geqslant \mu k \ . \tag{7.47}$$

Definition 7.21 (**Configuration** $(\diamond\mathbf{6})$)**.** *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{6})(\delta, \tilde{\varepsilon}, d', \mu, \gamma', h_2)$ *if the following conditions are met.*

The vertex sets V_0, V_1 witness Preconfiguration $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ or Preconfiguration $(\mathbf{exp})(\delta)$ and either Preconfiguration $(\heartsuit\mathbf{1})(\gamma', h_2)$ or Preconfiguration $(\heartsuit\mathbf{2})(h_2)$. There exist non-empty sets $V_2, V_3 \subseteq \mathfrak{P}_1$ such that

$$\mathrm{deg^{min}}_G(V_1, V_2) \geqslant \delta k \ , \tag{7.48}$$

$$\mathrm{deg^{min}}_G(V_2, V_1) \geqslant \delta k \ , \tag{7.49}$$

$$\mathrm{deg^{min}}_{G_{\exp}}(V_2, V_3) \geqslant \delta k \ , \text{and} \tag{7.50}$$

$$\mathrm{deg^{min}}_{G_{\exp}}(V_3, V_2) \geqslant \delta k \ . \tag{7.51}$$

Definition 7.22 (**Configuration** $(\diamond\mathbf{7})$)**.** *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{7})(\delta, \rho', \tilde{\varepsilon}, d', \mu, \gamma', h_2)$ *if the following conditions are met.*

The sets V_0, V_1 witness Preconfiguration $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ and either Preconfiguration $(\heartsuit\mathbf{1})(\gamma', h_2)$ or Preconfiguration $(\heartsuit\mathbf{2})(h_2)$. There exist non-empty sets $V_2 \subseteq \mathfrak{A}^{\restriction 1} \setminus \bar{V}$ and $V_3 \subseteq \mathfrak{P}_1$ such that

$$\mathrm{deg^{min}}_G(V_1, V_2) \geqslant \delta k \ , \tag{7.52}$$

$$\mathrm{deg^{min}}_G(V_2, V_1) \geqslant \delta k \ , \tag{7.53}$$

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}(V_2, \mathfrak{P}_1 \setminus V_3) < \rho' k \ \text{and} \tag{7.54}$$

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_3, V_2) \geqslant \delta k \ . \tag{7.55}$$

Definition 7.23 (**Configuration** $(\diamond\mathbf{8})$)**.** *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{8})(\delta, \rho', \varepsilon_1, \varepsilon_2, d_1, d_2, \mu_1, \mu_2, h_1, h_2)$ *if the following conditions are met.*

The vertex sets V_0, V_1 witness Preconfiguration $(\mathbf{reg})(\varepsilon_2, d_2, \mu_2)$ and Preconfiguration $(\heartsuit\mathbf{2})(h_2)$. There exist non-empty sets $V_2 \subseteq \mathfrak{P}_0$, $V_3, V_4 \subseteq \mathfrak{P}_1$, $V_3 \subseteq \mathfrak{A} \setminus \bar{V}$, and an $(\varepsilon_1, d_1, \mu_1 k)$-semiregular

matching $\mathcal{N}$ absorbed by $(\mathcal{M}_A \cup \mathcal{M}_B) \setminus \mathcal{N}_{\mathfrak{A}}$, $V(\mathcal{N}) \subseteq \mathfrak{P}_1 \setminus V_3$ such that

$$\deg^{\min}{}_G(V_1, V_2) \geqslant \delta k \;, \tag{7.56}$$
$$\deg^{\min}{}_G(V_2, V_1) \geqslant \delta k \;, \tag{7.57}$$
$$\deg^{\min}{}_{G_\nabla}(V_2, V_3) \geqslant \delta k \;, \tag{7.58}$$
$$\deg^{\min}{}_{G_\nabla}(V_3, V_2) \geqslant \delta k \;, \tag{7.59}$$
$$\deg^{\max}{}_{G_{\mathcal{D}}}(V_3, \mathfrak{P}_1 \setminus V_4) < \rho' k \;, \tag{7.60}$$
$$\deg^{\min}{}_{G_{\mathcal{D}}}(V_4, V_3) \geqslant \delta k \;, \textit{ and} \tag{7.61}$$
$$\deg_{G_{\mathcal{D}}}(v, V_3) + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) \geqslant h_1 \textit{ for each } v \in V_2. \tag{7.62}$$

Definition 7.24 (**Configuration** $(\diamond\mathbf{9})$)**.** *Suppose that we are in Settings 7.4, and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{9})(\delta, \gamma', h_1, h_2, \varepsilon_1, d_1, \mu_1, \varepsilon_2, d_2, \mu_2)$ *if the following conditions are met.*

The sets V_0, V_1 together with the $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{F}'$ witness Preconfiguration $(\heartsuit\mathbf{1})(\gamma', h_2)$. There exists an $(\varepsilon_1, d_1, \mu_1 k)$-semiregular matching $\mathcal{N}$ absorbed by $\mathcal{M}_A \cup \mathcal{M}_B$, $V(\mathcal{N}) \subseteq \mathfrak{P}_1$. Further, there is a family $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j \in \mathcal{Y}}$ as in Preconfiguration $(\mathbf{reg})(\varepsilon_2, d_2, \mu_2)$. There is a set $V_2 \subseteq V(\mathcal{N}) \setminus \bigcup \mathcal{F}' \subseteq \bigcup \mathbf{V}$ with the following properties:

$$\deg^{\min}{}_{G_{\mathcal{D}}}(V_1, V_2) \geqslant h_1 \;, \tag{7.63}$$
$$\deg^{\min}{}_{G_{\mathcal{D}}}(V_2, V_1) \geqslant \delta k \;. \tag{7.64}$$

Our last configuration, Configuration $(\diamond\mathbf{10})$, will lead to an embedding very similar to the one in the dense case (treated in [PS12]; see Section 8.1.5). In order to be able to formalize the configuration we need a preliminary definition. We shall generalize the standard concept of a regularity graph (in the context of regular partitions and Szemerédi's Regularity Lemma) to graphs with clusters whose sizes are only bounded from below.

Definition 7.25 (**$(\varepsilon, d, \ell_1, \ell_2)$-regularized graph**)**.** *Let G be a graph, and let $\mathcal{V}$ be an ℓ_1-ensemble that partitions $V(G)$. Suppose that $G[X]$ is empty for each $X \in \mathcal{V}$ and suppose $G[X, Y]$ is ε-regular and of density either 0 or at least d for each $X, Y \in \mathcal{V}$. Further suppose that for all $X \in \mathcal{V}$ it holds that $|\bigcup \mathrm{N}_G(X)| \leqslant \ell_2$. Then we say that $(G, \mathcal{V})$ is an $(\varepsilon, d, \ell_1, \ell_2)$-*regularized graph.

A semiregular matching $\mathcal{M}$ of G is consistent *with $(G, \mathcal{V})$ if $\mathcal{V}(\mathcal{M}) \subseteq \mathcal{V}$.*

Definition 7.26 (**Configuration** $(\diamond\mathbf{10})(\tilde{\varepsilon}, d', \ell_1, \ell_2, \eta')$)**.** *Assume Setting 7.4. The graph G contains an $(\tilde{\varepsilon}, d', \ell_1, \ell_2)$-regularized graph $(\tilde{G}, \mathcal{V})$ and there is a $(\tilde{\varepsilon}, d', \ell_1)$-semiregular matching $\mathcal{M}$ consistent with $(\tilde{G}, \mathcal{V})$. There are a family $\mathcal{L}^* \subseteq \mathcal{V}$ and distinct clusters $A, B \in \mathcal{V}$ with*

(a) $E(\tilde{G}[A, B]) \neq \emptyset$,

(b) $\deg_{\tilde{G}}\left(v, V(\mathcal{M}) \cup \bigcup \mathcal{L}^*\right) \geqslant (1 + \eta')k$ *for all but at most $\tilde{\varepsilon}|A|$ vertices $v \in A$ and for all but at most $\tilde{\varepsilon}|B|$ vertices $v \in B$, and*

(c) for each $X \in \mathcal{L}^*$ we have $\deg_{\tilde{G}}(v) \geqslant (1 + \eta')k$ for all but at most $\tilde{\varepsilon}|X|$ vertices $v \in X$.

7.5 The role of random splitting

The random splitting as introduced in Setting 7.7 is used in Configurations ($\diamond$**6**)–($\diamond$**9**); the set $\mathfrak{P}_0$ will host the cut-vertices $W_A \cup W_B$, the set $\mathfrak{P}_1$ will host the internal shrubs, and the set $\mathfrak{P}_2$ will (essentially) host the end shrubs of a (τk)-fine partition of $T_{\triangleright \mathrm{T}1.3}$.

The need for introducing the random splitting is dictated by Configurations ($\diamond$**6**)–($\diamond$**9**). To see this, let us try to follow the embedding plan from, for example, Section 8.1.2 without the random splitting, i.e., dropping the conditions $\subseteq \mathfrak{P}_0$, $\subseteq \mathfrak{P}_1$, $\subseteq \mathfrak{P}_2$ from Definitions 7.17–7.22. Then the sets V_2 and V_3 in Figure 8.2, which will host the internal shrubs, may interfere with V_0 and V_1 primarily designated for W_A and W_B. In particular, the conditions on degrees between V_0 and V_1 given by (7.45)–(7.46) in Definition 7.19, or given by the super-regularity in Definition 7.20 (in which $\beta_{\triangleright \mathrm{D}7.19} > 0$, or $d'_{\triangleright \mathrm{D}7.20}\mu_{\triangleright \mathrm{D}7.20} > 0$ are tiny) need not be sufficient for embedding greedily all the cut-vertices and all the internal shrubs of $T_{\triangleright \mathrm{T}1.3}$. It should be noted that this problem occurs even in Preconfiguration (**exp**), i.e., the expanding property does not add enough strength to the minimum degree conditions due to the same peculiarity as in Figure 4.2. Restricting V_0 and V_1 to host only the cut-vertices (only $O(1/\tau) = o(k)$ of them in total, cf. Definition 3.1(c)), resolves the problem.

The above justifies the distinction between the space $\mathfrak{P}_0$ for embedding the cut-vertices and the space $\mathfrak{P}_1 \cup \mathfrak{P}_2$ for embedding the shrubs. There are some other approaches which do not need to further split $\mathfrak{P}_1 \cup \mathfrak{P}_2$ but doing so seems to be the most convenient.

7.6 Cleaning

This section contains five "cleaning lemmas" (Lemma 7.27–Lemma 7.31). The basic setting of all these lemmas is the same. There is a system of vertex sets and some density assumptions on edges between certain sets of this system. The assertion the is that a small number of vertices can be discarded from the sets so that some conditions on the minimum degree are fullfilled. While the cleaning strategy is simply discarding the vertices which violate these minimum degree conditions the analysis of the outcome is non-trivial and employs amortized analysis. The simplest application of such an approach was the proof of Lemma 7.5 above.

Lemmas 7.27–7.31 are used to get the structures required by (pre-)configurations introduced in Section 7.4, based on rough structures found in Lemma 6.1.

The first lemma will be used to obtain preconfiguration ($\clubsuit$) in certain situations.

Lemma 7.27. *Let $\psi \in (0,1)$, and $\Gamma, \Omega \geqslant 1$ be arbitrary. Let P and Q be two disjoint vertex sets in a graph G. Assume that $Y \subseteq V(G)$ is given. We assume that*

$$\deg^{\min}(P, Q) \geqslant \Omega k \,, \tag{7.65}$$

and $\deg^{\max}(Q) \leqslant \Gamma k$. Then there exist sets $P' \subseteq P$, $Q' \subseteq Q \setminus Y$ and $Q'' \subseteq Q'$ such that the following holds.

(a) $\deg^{\min}(P', Q'') \geqslant \frac{\psi^3 \Omega}{4\Gamma^2} k$,

(b) $\deg^{\max}(Q', P \setminus P') < \psi k$,

(c) $\deg^{\max}(Q'', Q \setminus Q') < \psi k$, *and*

(d) $e(P', Q'') \geqslant (1-\psi) e(P, Q) - \frac{2|Y \cap Q| \Gamma^2}{\psi} k$.

Proof. Initially, set $P' := P$, $Q' =: Q \setminus Y$ and $Q'' := Q' \setminus Y$. We shall sequentially discard from the sets P', Q' and Q'' those vertices which violate any of the properties (a)–(c). Further, if a vertex $v \in Q$ is removed from Q' then we remove it from the set Q'' as well. This way, we have $Q'' \subseteq Q'$ in each step. After this sequential cleaning procedure finishes it only remains to establish (d).

First, observe that the way we constructed P' ensures that

$$e(P \setminus P', Q'') \leqslant \frac{\psi^3}{4\Gamma^2} e(P, Q) \; . \tag{7.66}$$

Let $Q^b \subseteq Q \setminus Q'$ be the set of the vertices removed because of condition (b). For a vertex $u \in P \setminus P'$, we write Q''_u for the set Q'' just before the moment when u was removed from P'. Likewise, we define the sets P'_v, Q'_v, Q''_v for each $v \in Q \setminus Q''$. For $u \in P \setminus P'$ let $f(u) := \deg(u, Q''_u)$, for $v \in Q \setminus (Q' \cup Y)$ let $g(v) := \deg(v, P \setminus P'_v)$, and for $w \in Q' \setminus Q''$ let $h(w) := \deg(w, Q \setminus Q'_w)$. Observe that $\sum_{u \in P \setminus P'} f(u) \geqslant \sum_{v \in Q^b} g(v)$. Indeed, at the moment when $v \in Q$ is removed from Q', the $g(v)$ edges that v sends to the set $P \setminus P'_v$ are counted in $\sum_{w \in \mathrm{N}(v) \cap (P \setminus P')} f(w)$. We therefore have

$$\frac{\psi^3}{4\Gamma^2} e(P, Q) \geqslant \frac{\psi^3}{4\Gamma^2} \sum_{u \in P \setminus P'} \deg(u, Q) \geqslant \sum_{u \in P \setminus P'} f(u) \geqslant \sum_{v \in Q^b} g(v) \geqslant |Q^b| \psi k \; ,$$

and consequently,

$$|Q^b| \leqslant \frac{\psi^2}{4\Gamma^2 k} e(P, Q) \; . \tag{7.67}$$

We also have

$$|Q' \setminus Q''| \psi k \leqslant \sum_{w \in Q' \setminus Q''} h(w) \leqslant |Q^b \cup (Y \cap Q)| \Gamma k \overset{(7.67)}{\leqslant} \frac{\psi^2}{4\Gamma} e(P, Q) + |Y \cap Q| \Gamma k \; . \tag{7.68}$$

Finally, we can lower-bound $e(P', Q'')$ as follows.

$$\begin{aligned}
e(P', Q'') &\geqslant e(P, Q) - e(P \setminus P', Q'') - |Y \cap Q| \Gamma k - |Q^b| \Gamma k - |Q' \setminus Q''| \Gamma k \\
\text{(by (7.66), (7.67), (7.68))} \quad &\geqslant e(P, Q) \Big(1 - \frac{\psi^3}{4\Gamma^2} - \frac{\psi^2}{4\Gamma} - \frac{\psi}{4}\Big) - |Y \cap Q| \Big(\frac{\Gamma^2 k}{\psi} + \Gamma k\Big) \\
&\geqslant (1-\psi) e(P, Q) - \frac{2}{\psi} |Y \cap Q| \Gamma^2 k \; .
\end{aligned}$$

□

The purpose of the lemmas below (Lemmas 7.28–7.31) is to distill vertex-sets for configurations ($\diamond$**2**)-($\diamond$**10**). They will be applied in Lemmas 7.33, 7.34, 7.35. This is the final “cleaning step” on our way to the proof of Theorem 1.3 — the outputs of these lemmas can by used for a vertex-by-vertex embedding of any tree $T \in \mathbf{trees}(k)$ (although the corresponding embedding procedures given in Section 8 are quite complex).

The first two of these cleaning lemmas (Lemmas 7.28 and 7.29) are suited when the set $\mathbf{\Psi}$ of vertices of huge degrees (cf. Setting 7.4) needs to be considered.

For the following lemma, recall that we defined $[r]$ as the set of the first r natural numbers, *not* including 0.

Lemma 7.28. *For all $r, \Omega^*, \Omega^{**} \in \mathbb{N}$, and $\delta, \gamma, \eta \in (0,1)$, with $\left(\frac{3\Omega^*}{\gamma}\right)^r \delta < \eta/10$, and $\Omega^{**} > 1000$ the following holds. Suppose there are vertex sets $X_0, X_1, \ldots, X_r$ and Y of an n-vertex graph G such that*

1. *$|Y| < \eta n/(4\Omega^*)$,*
2. *$e(X_0, X_1) \geqslant \eta k n$,*
3. *$\deg^{\min}(X_0, X_1) \geqslant \Omega^{**} k$,*
4. *$\deg^{\min}(X_i, X_{i+1}) \geqslant \gamma k$ for all $i \in [r-1]$, and*
5. *$\deg^{\max}\left(Y \cup \bigcup_{i\in[r]} X_i\right) \leqslant \Omega^* k$.*

Then there are sets $X'_i \subseteq X_i$ for $i = 0, 1, \ldots, r$ such that

(a) $X'_1 \cap Y = \emptyset$,

(b) $\deg^{\min}(X'_i, X'_{i-1}) \geqslant \delta k$ for all $i \in [r]$,

(c) $\deg^{\max}(X'_i, X_{i+1} \setminus X'_{i+1}) < \gamma k/2$ for all $i \in [r-1]$,

(d) $\deg^{\min}(X'_0, X'_1) \geqslant \sqrt{\Omega^{**}} k$, and

(e) $e(X'_0, X'_1) \geqslant \eta k n/2$, in particular $X'_0 \neq \emptyset$.

Proof. In the formulae below we refer to hypotheses of the lemma as “1.”–“5.”.

Set $X'_1 := X_1 \setminus Y$. For $i = 0, 2, 3, 4, \ldots, r$, set $X'_i := X_i$. Discard sequentially from X'_i any vertex that violates any of the Properties (b)–(d). Properties (a)–(d) are trivially satisfied when the procedure terminates. To show that Property (e) holds at this point, we bound the number of edges from $e(X_0, X_1)$ that are incident with $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$ in an amortized way.

For $i \in \{0, \ldots, r\}$ and for $v \in X_i \setminus X'_i$ we write

$$\begin{aligned} f_i(v) &:= \deg\left(v, X_{i+1}(v) \setminus X'_{i+1}(v)\right), \\ g_i(v) &:= \deg\left(v, X'_{i-1}(v)\right), \text{ and} \\ h_i(v) &:= \deg\left(v, X'_{i+1}(v)\right). \end{aligned}$$

where the sets $X'_{i-1}(v), X'_i(v), X'_{i+1}(v)$ above refer to the moment when v is removed from X'_i (we do not define $f_i(v)$ and $h_i(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

For $i \in [r]$ let $X_i^{\mathfrak{b}}$ denote the vertices in $X_i \setminus X'_i$ that were removed from X'_i because of violating Property (b). Then for a given $i \in [r]$ we have that

$$\sum_{v \in X_i^{\mathfrak{b}}} g_i(v) < \delta k n. \tag{7.69}$$

For $i = 1, \ldots, r-1$ let $X_i^{\mathfrak{c}}$ denote the vertices in $X_i \setminus X'_i$ that violated Property (c). Set $X_r^{\mathfrak{c}} := \emptyset$. For a given $i \in [r-1]$ we have

$$|X_i^{\mathfrak{c}}| \cdot \gamma k/2 \leqslant \sum_{v \in X_i^{\mathfrak{c}}} f_i(v) \leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) \overset{5.,(7.69)}{<} \delta k n + |X_{i+1}^{\mathfrak{c}}| \cdot \Omega^* k \ , \tag{7.70}$$

as $X_i \setminus X'_i = X_i^{\mathfrak{b}} \cup X_i^{\mathfrak{c}}$, for $i = 2, \ldots, r$. Using (7.70) for $j = 0, \ldots, r-1$, we inductively deduce that

$$|X_{r-j}^{\mathfrak{c}}| \frac{\gamma}{2} \leqslant \sum_{i=0}^{j-1} \left(\frac{2\Omega^*}{\gamma} \right)^i \delta n \ . \tag{7.71}$$

(The left-hand side is zero for $j = 0$.) The bound (7.71) for $j = r-1$ gives

$$|X_1^{\mathfrak{c}}| \leqslant \frac{2}{\gamma} \cdot \sum_{i=0}^{r-2} \left(\frac{2\Omega^*}{\gamma} \right)^i \delta n \leqslant \frac{2(2\Omega^*)^{r-1}}{\gamma^r} \delta n \ . \tag{7.72}$$

Therefore,

$$e(X_0, Y \cup X_1^{\mathfrak{c}}) \leqslant |Y \cup X_1^{\mathfrak{c}}| \cdot \Omega^* k \overset{(7.72),1.}{\leqslant} \frac{\eta k n}{4} + \left(\frac{2\Omega^*}{\gamma} \right)^r \delta k n \ . \tag{7.73}$$

For any vertex $v \in X_0 \setminus X'_0$ we have $h_0(v) < \sqrt{\Omega^{**}} k$, and at the same time by Hypothesis 3. we have $\deg(v, X_1) \geqslant \Omega^{**} k$. So,

$$\sum_{v \in X_0 \setminus X'_0} h_0(v) \leqslant \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} \ . \tag{7.74}$$

We have

$$e(X'_0, X'_1) \geqslant e(X_0, X_1) - e(X_0, Y \cup X_1^{\mathfrak{c}}) - \sum_{v \in X_0 \setminus X'_0} h_0(v) - \sum_{v \in X_1^{\mathfrak{b}}} g_1(v) \ .$$

(It requires a minute of meditation to see that edges between $X_0 \setminus X'_0$ and $X_1^{\mathfrak{b}}$ are indeed not counted on the right-hand side.) Therefore,

$$\begin{aligned} e(X'_0, X'_1) &\geqslant e(X_0, X_1) - e(X_0, Y \cup X_1^{\mathfrak{c}}) - \sum_{v \in X_0 \setminus X'_0} h_0(v) - \sum_{v \in X_1^{\mathfrak{b}}} g_1(v) \\ \text{(by (7.69), (7.73), (7.74))} \quad &\geqslant e(X_0, X_1) - \frac{\eta k n}{4} - \left(\frac{2\Omega^*}{\gamma} \right)^r \delta k n - \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} - \delta k n \\ \text{(by 2.)} \quad &\geqslant \eta k/2 \ , \end{aligned}$$

proving Property (e). $\square$

Lemma 7.29. *Let $\delta, \eta, \Omega^*, \Omega^{**}, h > 0$, let G be an n-vertex graph, let $X_0, X_1, Y \subseteq V(G)$, and let $\mathcal{C}$ be a system of subsets of $V(G)$ such that*

1. $20(\delta + \frac{2}{\sqrt{\Omega^{**}}}) < \eta$,
2. $2kn \geqslant e(X_0, X_1) \geqslant \eta kn$,
3. $\deg^{\min}(X_0, X_1) \geqslant \Omega^{**}k$,
4. $\deg^{\max}(X_1) \leqslant \Omega^* k$,
5. $|Y| < \eta n/(4\Omega^*)$, *and*
6. $10h|\mathcal{C}|\Omega^* < \eta n$.

Then there are sets $X'_0 \subseteq X_0$ and $X'_1 \subseteq X_1 \setminus Y$ such that

a) $\deg^{\min}(X'_0, X'_1) \geqslant \sqrt{\Omega^{**}}k$,

b) $\deg^{\min}(X'_1, X'_0) \geqslant \delta k$,

c) for all $C \in \mathcal{C}$, either $X'_1 \cap C = \emptyset$, or $|X'_1 \cap C| \geqslant h$, and

d) $e(X'_0, X'_1) \geqslant \eta kn/2$.

Proof. Set $X'_0 := X_0$ and $X'_1 := X_1 \setminus Y$ and discard sequentially from X'_0, any vertex violating Property a). Further, we discard from X'_1 any vertex violating Property b), or any $C \in \mathcal{C}$ violating c). When the process ends, we verify Property d) by bounding the number of edges in $e(X_0, X_1)$ incident with $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$. Given Assumption 2, and since by Assumption 5 there are at most $\frac{1}{4}\eta kn$ edges incident with $Y \cap X_1$ it suffices to prove that

$$e(X_0, X_1) - e(X'_0, X'_1) - e(Y \cap X_1, X_0) < \frac{\eta kn}{4} \,. \tag{7.75}$$

Denote by X_1^b the set of vertices in $X_1 \setminus (Y \cup X'_1)$ that violated Property b), and by X_1^c the set of vertices in $X_1 \setminus (Y \cup X'_1)$ that violated Property c). For a vertex $v \in X_1 \setminus (Y \cup X'_1)$, let $g(v)$ denote the number $\deg(v, X'_0)$ at the very time when v is removed from X'_1. Analogously we define $f(v)$, for $v \in X_0 \setminus X'_0$, as $\deg(v, X'_1)$ where the set X'_1 is considered at the point of removal of v. We have $\sum_{v \in X_1^b} g(v) < \delta kn$, $\sum_{v \in X_1^c} g(v) \leqslant |X_1^c|\Omega^* k < h|\mathcal{C}| \cdot \Omega^* k$, and

$$\sum_{v \in X_0 \setminus X'_0} f(v) \leqslant \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} \overset{2.}{\leqslant} \frac{2}{\sqrt{\Omega^{**}}} kn \,.$$

Thus,

$$\begin{aligned}
e(X_0, X_1) - e(X_0', X_1') - e(Y \cap X_1, X_0) \\
&= \sum_{v \in X_1^b} g(v) + \sum_{v \in X_1^C} g(v) + \sum_{v \in X_0 \setminus X_0'} f(v) \\
&< \left(\delta + \frac{2}{\sqrt{\Omega^{**}}}\right) kn + h|\mathcal{C}|\Omega^* k \\
\text{(by 1. and 6.)} \quad &< \frac{\eta k n}{4} \,.
\end{aligned}$$

establishing (7.75). □

The next two lemmas (Lemmas 7.30 and 7.31) deal with cleaning outside the set of huge degree vertices $\mathbf{\Psi}$.

Lemma 7.30. *For all $r, \Omega \in \mathbb{N}$, $r \geqslant 2$ and all $\gamma, \delta, \eta > 0$ such that*

$$\left(\frac{8\Omega}{\gamma}\right)^r \delta \leqslant \frac{\eta}{10} \tag{7.76}$$

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \ldots, X_r \subseteq V$, where V is a set of n vertices. Suppose that edge sets $E_1, \ldots, E_r$ are given on V. The expressions deg_i, $\mathrm{deg^{max}}_i$, $\mathrm{deg^{min}}_i$, and e_i below refer to the edge set E_i. Suppose that the following properties are fulfilled

1. *$|Y| < \delta n$,*
2. *$\mathrm{e}_1(X_0, X_1) \geqslant \eta k n$,*
3. *for all $i \in [r-1]$ we have $\mathrm{deg^{min}}_{i+1}(X_i \setminus Y, X_{i+1}) \geqslant \gamma k$,*
4. *for all $i \in \{0, \ldots, r-1\}$, we have $\mathrm{deg^{max}}_{i+1}(X_i) \leqslant \Omega k$, and $\mathrm{deg^{max}}_{i+1}(X_{i+1}) \leqslant \Omega k$.*

Then there are sets $X_i' \subseteq X_i \setminus Y$ $(i = 0, \ldots, r)$ satisfying the following.

a) For all $i \in [r]$ and we have $\mathrm{deg^{min}}_i(X_i', X_{i-1}') \geqslant \delta k$,

b) for all $i \in [r-1]$ we have $\mathrm{deg^{max}}_{i+1}(X_i', X_{i+1} \setminus X_{i+1}') < \gamma k/2$,

c) $\mathrm{deg^{min}}_1(X_0', X_1') \geqslant \delta k$, and

d) $\mathrm{e}_1(X_0', X_1') \geqslant \eta k n/2$

Proof. We proceed similarly as in the proof of Lemma 7.28. Set $X_i' := X_i \setminus Y$ for each $i = 0, \ldots, r$. Discard sequentially from X_i' any vertex that violates Property a) or b), or c). When the procedure terminates, we certainly have that a)–c) hold. We then show that Property d) holds by bounding the

number of edges from $e_1(X_0, X_1)$ that are incident with $X_0 \setminus X_0'$ or with $X_1 \setminus X_1'$. For $i \in \{0, \ldots, r\}$ and for $v \in X_i \setminus X_i'$ we write

$$\begin{aligned} f_{i+1}(v) &:= \deg_{i+1}(v, X_{i+1} \setminus X_{i+1}') \ , \\ g_i(v) &:= \deg_i(v, X_{i-1}') \ , \text{ and} \\ h(v) &:= \deg_1(v, X_1') \ , \end{aligned}$$

where the sets X_1', X_{i-1}' and X_{i+1}' above refer to the moment[18] when v is removed from X_i' or from X_1' (we do not define $f_{i+1}(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

Let $X_i^a \subseteq X_i$, $X_i^b \subseteq X_i$ for $i \in [r-1]$ be the sets of vertices removed from X_i' because of Property a) and b), respectively. Set $X_r^a := X_r \setminus X_r'$ and $X_0^c := X_0 \setminus X_0'$. We have for each $i \in [r]$,

$$\sum_{v \in X_i^a} g_i(v) < \delta k n \ . \tag{7.77}$$

Also, note that we have

$$\sum_{v \in X_0^c} h(v) \leqslant \delta k n \ . \tag{7.78}$$

We set $X_r^b := \emptyset$. For a given $i \in [r-1]$ we have

$$\begin{aligned} |X_i^b| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^b} f_{i+1}(v) \\ &\leqslant \sum_{v \in X_{i+1} \setminus X_{i+1}'} g_{i+1}(v) \\ \text{(by 4., (7.77))} \quad &\leqslant \delta k n + |X_{i+1}^b| \Omega k \ , \end{aligned} \tag{7.79}$$

as $X_i \setminus X_i' \subseteq X_i^a \cup X_i^b \cup Y$, for $i = 2, \ldots, r$. Using (7.79), we deduce inductively that

$$|X_{r-j}^b| \leqslant \left(\frac{8\Omega}{\gamma} \right)^j \delta n \ , \tag{7.80}$$

for $j = 0, \ldots, r-1$. (The left-hand side is zero for $j = 0$.) Therefore,

$$\begin{aligned} e_1(X_0', X_1') &\geqslant e_1(X_0, X_1) - (|Y| + |X_1^b|)\Omega k - \sum_{v \in X_1^a} g_1(v) - \sum_{v \in X_0^c} h(v) \\ \text{(by 2, (7.80), (7.77), (7.78))} \quad &\geqslant \eta k n - \left(\frac{8\Omega}{\gamma} \right)^r \delta k n - 2\delta k n \\ &\geqslant \frac{\eta}{2} k n \ , \end{aligned}$$

establishing Property d). □

[18] if $v \in Y$ then this moment is the zero-th step

Lemma 7.31. *For all $r, \Omega \in \mathbb{N}$, $r \geqslant 2$ and all $\gamma, \eta, \delta, \varepsilon, \mu, d > 0$ with*

$$20\varepsilon < d \quad \text{and} \quad \left(\frac{8\Omega}{\gamma}\right)^r \delta \leqslant \frac{\eta}{30} \tag{7.81}$$

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \ldots, X_r \subseteq V$, where V is a set of n vertices. Let $P_i^{(1)}, \ldots, P_i^{(p)}$ partition X_i, for $i = 0, 1$. Suppose that edge sets $E_1, E_2, E_3, \ldots, E_r$ are given on V. The expressions $\deg_i$, ${\deg^{\max}}_i$, and ${\deg^{\min}}_i$ below refer to the edge set E_i. Suppose that

1. *$|Y| < \delta n$,*
2. *$|X_1| \geqslant \eta n$,*
3. *for all $i \in [r-1]$ we have ${\deg^{\min}}_{i+1}(X_i \setminus Y, X_{i+1}) \geqslant \gamma k$,*
4. *the family $\left\{(P_0^{(j)}, P_1^{(j)})\right\}_{j\in[p]}$ is an $(\varepsilon, d, \mu k)$-semiregular matching with respect to the edge set E_1, and*
5. *for all $i \in \{0, \ldots, r-1\}$, ${\deg^{\max}}_{i+1}(X_{i+1}) \leqslant \Omega k$, and (when $i \neq r$) ${\deg^{\max}}_{i+1}(X_i) \leqslant \Omega k$.*

Then there is a non-empty family $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j\in\mathcal{Y}}$ of vertex-disjoint $(4\varepsilon, \frac{d}{4})$-super-regular pairs with respect to E_1, with

a) $|Q_0^{(j)}|, |Q_1^{(j)}| \geqslant \frac{\mu k}{2}$ for each $j \in \mathcal{Y}$,

and sets $X_0' := \bigcup Q_0^{(j)} \subseteq X_0 \setminus Y$, $X_1' := \bigcup Q_1^{(j)} \subseteq X_1 \setminus Y$, $X_i' \subseteq X_i \setminus Y$ $(i = 2, \ldots, r)$ such that

b) for all $i \in [r-1]$ we have ${\deg^{\min}}_{i+1}(X_{i+1}', X_i') \geqslant \delta k$, and

c) for all $i \in [r-1]$, we have ${\deg^{\max}}_{i+1}(X_i', X_{i+1} \setminus X_{i+1}') < \gamma k/2$.

Proof. Initially, set $\mathcal{J} := \emptyset$ and $X_i' := X_i \setminus Y$ for each $i = 0, \ldots, r$. Discard sequentially from X_i' any vertex that violates any of the Properties b) or c). We would like to keep track of these vertices and therefore we call $X_i^b, X_i^c \subseteq X_i$ the sets of vertices removed from X_i' because of Property b), and c), respectively. Further, for $i = 0, 1$ and for $j \in [p]$ remove any vertex $v \in X_i' \cap P_i^{(j)}$ from X_i' if

$$\deg_1(v, X_{1-i}' \cap P_{1-i}^{(j)}) \leqslant \frac{d|P_{1-i}^{(j)}|}{4}\ . \tag{7.82}$$

For $i = 0, 1$, let X_i^a be the set of those vertices of X_i that were removed because of (7.82).

Last, if for some $j \in [p]$ we have $|P_0^{(j)} \cap Y| > \frac{|P_0^{(j)}|}{4}$ or $|P_1^{(j)} \cap (Y \cup X_1^c)| > \frac{|P_1^{(j)}|}{4}$ we remove simultaneously the sets $P_0^{(j)}$ and $P_1^{(j)}$ entirely from X_0' and X_1', i.e., we set $X_0' := X_0' \setminus P_0^{(j)}$ and $X_1' := X_1' \setminus P_1^{(j)}$. We also add the index j to the set $\mathcal{J}$ in this case.

When the procedure terminates define $\mathcal{Y} := [p] \setminus \mathcal{J}$, and for $j \in \mathcal{Y}$ set $(Q_0^{(j)}, Q_1^{(j)}) := (P_0^{(j)} \cap X_0', P_1^{(j)} \cap X_1')$. The sets X_i' obviously satisfy Properties b)–c). We now turn to verifying Property a). This relies on the following claim.

Claim 7.31.1. If $j \in [p] \setminus \mathcal{J}$ then $|P_0^{(j)} \cap X_0^a| \leqslant \frac{|P_0^{(j)}|}{4}$ and $|P_1^{(j)} \cap X_1^a| \leqslant \frac{|P_1^{(j)}|}{4}$.

Proof of Claim 7.31.1. Recall that E_1 is the relevant underlying edge set when working with the pairs $(P_0^{(j)}, P_1^{(j)})$. Also, recall that only vertices from $Y \cup X_0^a$ were removed from $P_0^{(j)}$ and only vertices from $Y \cup X_1^a \cup X_1^c$ were removed from $P_1^{(j)}$.

Since $j \notin \mathcal{J}$, the pair $(P_0^{(j)} \setminus Y, P_1^{(j)} \setminus (Y \cup X_1^c))$ is 2ε-regular of density at least $0.9d$ by Fact 2.7. Let

$$\begin{aligned} K_0 &:= \left\{ v \in P_0^{(j)} \setminus Y \ : \ \deg_1(v, P_1^{(j)} \setminus (Y \cup X_1^c)) < 0.8d|P_1^{(j)} \setminus (Y \cup X_1^b)| \right\} \text{, and} \\ K_1 &:= \left\{ v \in P_1^{(j)} \setminus (Y \cup X_1^c) \ : \ \deg_1(v, P_0^{(j)} \setminus Y) < 0.8d|P_0^{(j)} \setminus Y| \right\} . \end{aligned}$$

By Fact 2.8, we have $|K_0| \leqslant 2\varepsilon|P_0^{(j)} \setminus Y| \leqslant 0.1d|P_0^{(j)}|$ and $|K_1| \leqslant 0.1d|P_1^{(j)}|$. In particular, we have

$$\begin{aligned} {\deg^{\min}}_1(P_0^{(j)} \setminus (Y \cup K_0), P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)) &\geqslant 0.8d|P_1^{(j)} \setminus (Y \cup X_1^c)| - |K_1| \\ &\geqslant 0.8d \cdot 0.75|P_1^{(j)}| - 0.1d|P_1^{(j)}| \qquad (7.83) \\ &> 0.25d|P_1^{(j)}| \text{ , and} \\ {\deg^{\min}}_1(P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1), P_0^{(j)} \setminus (Y \cup K_0)) &\geqslant 0.8d|P_0^{(j)} \setminus Y| - |K_0| \\ &\geqslant 0.8d \cdot 0.75|P_0^{(j)}| - 0.1d|P_0^{(j)}| \qquad (7.84) \\ &> 0.25d|P_0^{(j)}| \ . \end{aligned}$$

Then (7.83) and (7.84) allow us to prove that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$. Indeed, assume inductively that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$ throughout the cleaning process until a certain step. Then (7.83) and (7.84) assert that no vertex outside of $P_0^{(j)} \setminus (Y \cup K_0)$ or of $P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)$ can be removed because of (7.82), proving the induction step. The claim follows. □

Putting together the definition of $\mathcal{J}$ (through which one controls the size of $P_i^{(j)} \cap (Y \cup X_i^c)$) and Claim 7.31.1 (which controls the size of $P_i^{(j)} \cap X_i^a$) we get for each $j \in \mathcal{Y}$ and $i = 0, 1$,

$$|Q_i^{(j)}| \geqslant \frac{|P_i^{(j)}|}{2} \geqslant \frac{\mu k}{2} \ .$$

Therefore, these pairs are 4ε-regular (cf. Fact 2.7). Last, we get the property of $(4\varepsilon, \frac{d}{4})$-super-regularity from the definition of X_i^c (cf. (7.82)). Thus, the pairs $(Q_0^{(j)}, Q_1^{(j)})$ are as required for Lemma 7.31 and satisfy its Property a).

The only thing we have to prove is that the set X_1' is nonempty. By the definition, for each $j \in \mathcal{J}$, we either have $|P_1^{(j)}| \leqslant 4(|(Y \cup X_1^c) \cap P_1^{(j)}|)$ or $|P_0^{(j)}| \leqslant 4|Y \cap P_0^{(j)}|$. We use that that $|P_0^{(j)}| = |P_1^{(j)}|$ to see that

$$\left| \bigcup_{\mathcal{J}} P_1^{(j)} \right| \leqslant 4(|Y| + |X_1^c|) \ . \qquad (7.85)$$

For $i \in \{1, \dots, r\}$ and for $v \in X_i \setminus X'_i$ write

$$
\begin{aligned}
f_{i+1}(v) &:= \deg_{i+1}(v, X_{i+1} \setminus X'_{i+1}) \text{ , and} \\
g_i(v) &:= \deg_i(v, X'_{i-1}) \ .
\end{aligned}
$$

where the sets X'_1, X'_{i-1} and X'_{i+1} above refer to the moment[19] when v is removed from X'_i (we do not define $f_{i+1}(v)$ for $i = r$).

Observe that for each $i \in \{2, \dots, r\}$, we have

$$
\sum_{v \in X_i^b} g_i(v) < \delta k n \ . \tag{7.86}
$$

We set $X_r^c := \emptyset$. For a given $i \in [r-1]$ we have

$$
\begin{aligned}
|X_i^{\mathcal{C}}| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^{\mathcal{C}}} f_{i+1}(v) \\
&\leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) & (7.87) \\
\text{(by 1. ,5. , (7.86))} \quad &< \delta k n + |X_{i+1}^{\mathcal{C}}| \Omega k, & (7.88)
\end{aligned}
$$

as $X_i \setminus X'_i \subseteq X_i^b \cup X_i^{\mathcal{C}} \cup Y$, for $i = 2, \dots, r$. Using (7.88), we deduce inductively that $|X_{r-j}^{\mathcal{C}}| \leqslant \left(\frac{8\Omega}{\gamma}\right)^j \delta n$ for $j = 1, 2, \dots, r-1$, and in particular that

$$
|X_1^{\mathcal{C}}| \leqslant \left(\frac{8\Omega}{\gamma}\right)^{r-1} \delta n \ . \tag{7.89}
$$

As $X_1^a = \emptyset$, we obtain that

$$
\begin{aligned}
|X'_1| &= \left| X_1 \setminus \left(\bigcup_{j \in \mathcal{J}} P_1^{(j)} \cup \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap (Y \cup X_1^a \cup X_1^c) \right) \right) \right| \\
\text{(by (7.85))} \quad &\geqslant |X_1| - 4(|Y| + |X_1^c|) - \left| \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap X_1^a \right) \right| \\
\text{(by 1., (7.81), (7.89))} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \left| \bigcup_{j \in \mathcal{Y}} (P_1^{(j)} \cap X_1^a) \right| \\
\text{(by Cl 7.31.1)} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \frac{|X_1|}{4} \\
\text{(by 2.)} \quad &> 0,
\end{aligned}
$$

as desired. $\square$

[19]if $v \in Y$ then this moment is the zero-th step

7.7 Obtaining a configuration

In this section we prove that the structure in the graph $G \in \mathbf{LKS}(n,k,\eta)$ guaranteed by Lemma 6.1 always leads to one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{10})$. We distinguish two cases. When the set $\mathbf{\Psi}$ of vertices of huge degree (coming from a sparse decomposition of G) sees many edges, then one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$ must occur (cf. Lemma 7.33). Otherwise, when the edges incident with $\mathbf{\Psi}$ can be neglected, we obtain one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$ (cf. Lemmas 7.34 and 7.35). How these configurations help in embedding the tree $T_{\triangleright\mathrm{T}1.3} \in \mathbf{trees}(k)$ will be shown in Section 8.

Lemmas 7.33, 7.34, and 7.35 are stated in the next section, and their proofs occupy Sections 7.7.3, 7.7.4, and 7.7.5, respectively. These results are put together in Lemma 7.32 of Section 7.7.1.

7.7.1 Statements of the results

We first state the main result of this section, Lemma 7.32. Its proof is given in Section 7.7.2.

Lemma 7.32. *Suppose we are in Settings 7.4 and 7.7. Further suppose that at least one of the cases* **(K1)** *or* **(K2)** *from Lemma 6.1 occurs in G (with $\mathcal{M}_{\mathrm{good}}$ as in Lemma 6.1* (h)*). Then one of the configurations*

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$,
- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$

occurs in G.

Lemma 7.32 will be proved in Section 7.7.2. The proof relies on Lemmas 7.33, 7.34 and 7.35 below. For an input graph $G_{\triangleright\mathrm{L}7.32}$ one of these lemmas is applied depending on the majority type

of "good" edges in $G_{\triangleright \mathrm{L}7.32}$. Observe that **(K1)** of Lemma 6.1 guarantees edges between $\mathbf{\Psi}$ and $\mathbb{XA} \cup \mathbb{XB}$, or between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$ either in $E(G_{\mathrm{exp}})$ or in $E(G_{\mathcal{D}})$. Lemma 7.33 is used if we find edges between $\mathbf{\Psi}$ and $\mathbb{XA} \cup \mathbb{XB}$. Lemma 7.34 is used if we find edges of $E(G_{\mathrm{exp}})$ between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$. The remaining case can be reduced to the setting of Lemma 7.35. Lemma 7.35 is also used to obtain a configuration if we are in case **(K2)** of Lemma 6.1.

Lemma 7.33. *Suppose we are in Setting 7.4. Assume that*

$$e_{G_{\nabla}}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \geqslant \frac{\eta^{13} kn}{10^{28} (\Omega^*)^3}. \tag{7.90}$$

Then G contains at least one of the configurations

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$, *or*
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$.

Lemma 7.34. *Suppose that we are in Setting 7.4 and Setting 7.7. If there exist two disjoint sets $\mathbb{YA}_1, \mathbb{YA}_2 \subseteq V(G)$ such that*

$$e_{G_{\mathrm{exp}}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho kn \;, \tag{7.91}$$

and either

$$\mathbb{YA}_1 \cup \mathbb{YA}_2 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}),\ \textit{or} \tag{7.92}$$

$$\mathbb{YA}_1 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F} \cup \mathtt{P}_2 \cup \mathtt{P}_3),\ \textit{and}\ \mathbb{YA}_2 \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}) \tag{7.93}$$

then G has configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Lemma 7.35. *Suppose that we are in Setting 7.4 and Setting 7.7. Let $\mathcal{D}_\nabla$ be as in Lemma 7.5. Suppose that there exists an $(\bar{\varepsilon}, \bar{d}, \beta k)$-semiregular matching $\mathcal{M}$, $V(\mathcal{M}) \subseteq \mathfrak{P}_0$, $|V(\mathcal{M})| \geqslant \frac{\rho n}{\Omega^*}$, with one of the following two sets of properties.*

(M1) *$\mathcal{M}$ is absorbed by $\mathcal{M}_{\mathrm{good}}$, $\bar{\varepsilon} := \frac{10^5\varepsilon'}{\eta^2}$, $\bar{d} := \frac{\gamma^2}{4}$, and $\beta := \frac{\eta^2\mathfrak{c}}{8\cdot 10^3 k}$.*

(M2) *$E(\mathcal{M}) \subseteq E(\mathcal{D}_\nabla)$, $\mathcal{M}$ is absorbed by $\mathcal{D}_\nabla$, $\bar{\varepsilon} := \pi$, $\bar{d} := \frac{\gamma^3\rho}{32\Omega^*}$, and $\beta := \frac{\widehat{\alpha}\rho}{\Omega^*}$.*

Suppose further that one of the following occurs.

(cA) $V(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$, *and we have for the set*

$$R := \mathbf{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right)$$

one of the following

(t1) $V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k)$,

(t2) $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathfrak{A}}$,

(t3) $V_1(\mathcal{M}) \subseteq R \setminus (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}})$, *or*

(t5) $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}}) \setminus (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}} \cup R)$.

(cB) $V_1(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \mathtt{P}_2 \cup \mathtt{P}_3 \cup \bar{V} \cup \mathbb{F})$ *and* $V_2(\mathcal{M}) \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$, *and we have*

(t1) $V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k)$,

(t2) $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathfrak{A}}$, *or*

(t3–5) $V_1(\mathcal{M}) \cap (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}}) = \emptyset$.

then at least one of the following configurations occurs:

- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$.

7.7.2 Proof of Lemma 7.32

In the proof, we distinguish different types of edges captured in cases **(K1)** and **(K2)**. If in case **(K1)** many of the captured edges from $\mathbb{XA}$ to $\mathbb{XA} \cup \mathbb{XB}$ are incident with $\mathbf{\Psi}$, we will get one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$ by employing Lemma 7.33. Otherwise, there must be many edges from $\mathbb{XA}$ to $\mathbb{XA} \cup \mathbb{XB}$ in the graph G_{exp}, or in $G_{\mathcal{D}}$. Lemma 7.34 shows that the former case leads to configuration $(\diamond\mathbf{6})$. We will reduce the latter case to the situation in Lemma 7.35 which gives one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$.

We use Lemma 7.35 to give one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$ also in case **(K2)**. [20]

Let us now turn to the details of the proof. If $e_G(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \geqslant \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}$ then we use Lemma 7.33 to obtain one of the configurations $(\diamond\mathbf{1})$–$(\diamond\mathbf{5})$, with the parameters as in the statement of Lemma 7.32.

[20] Actually, our proof of Lemma 7.35 implies that one does not get configuration $(\diamond\mathbf{9})$ in case **(K2)**; but this fact is never needed.

Thus, in the remainder of the proof we assume that

$$e_G(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) < \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}\ . \tag{7.94}$$

We now bound the size of the set $\mathtt{P}$. By Setting 7.4(9) we have that

$$|E(G) \setminus E(G_\nabla)| \leqslant 2\rho kn.$$

Plugging this into Lemma 7.9 we get $|L_\#| \leqslant \frac{40\rho n}{\eta}$, $|\mathbb{XA} \setminus \mathbb{YA}| \leqslant \frac{1200\rho n}{\eta^2}$, and $|(\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}| \leqslant \frac{1200\rho n}{\eta^2}$. Further, using (7.94), Lemma 7.9 also gives that $|V_{\not\leadsto\mathbf{\Psi}}| \leqslant \frac{\eta^{12}n}{10^{26}(\Omega^*)^3}$. It follows from Setting 7.4(8) that $|\mathtt{P}_{\mathfrak{A}}| \leqslant \gamma n$. Last, by Setting 7.4(7) we have $|\mathtt{P}_1| \leqslant 2\gamma n$. Thus,

$$\begin{aligned}
|\mathtt{P}| &\leqslant |\mathbb{XA} \setminus \mathbb{YA}| + |(\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}| + |V_{\not\leadsto\mathbf{\Psi}}| + |L_\#| + |\mathtt{P}_1| \\
&\quad + \left|\mathbf{shadow}_{G_{\mathcal{D}} \cup G_\nabla}(V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_{\mathfrak{A}} \cup \mathtt{P}_1, \frac{\eta^2 k}{10^5})\right| \\
&\overset{(7.3)}{\leqslant} \frac{2\eta^{10}n}{10^{21}(\Omega^*)^2}\ ,
\end{aligned} \tag{7.95}$$

where we used Fact 7.1 to bound the size of the shadows.

Let us first turn our attention to case **(K1)**. By Definition 7.6 we have $\mathbf{\Psi} \cap \mathfrak{P}_0 = \emptyset$. Therefore,

$$\begin{aligned}
e_{G_\nabla}\big(\mathbb{XA}^{\restriction 0} \setminus \mathtt{P}, (\mathbb{XA} \cup \mathbb{XB})^{\restriction 0} \setminus \mathtt{P}\big) &= e_{G_\nabla}\big((\mathbb{XA} \setminus (\mathbf{\Psi} \cup \mathtt{P}))^{\restriction 0}, (\mathbb{XA} \setminus (\mathbf{\Psi} \cup \mathtt{P}))^{\restriction 0} \cup (\mathbb{XB} \setminus \mathtt{P})^{\restriction 0}\big) \\
\text{(by Def 7.6 (7))} \quad &\geqslant \mathfrak{p}_0^2 \cdot e_{G_\nabla}\big(\mathbb{XA} \setminus (\mathbf{\Psi} \cup \mathtt{P}), (\mathbb{XA} \cup \mathbb{XB}) \setminus (\mathbf{\Psi} \cup \mathtt{P})\big) - k^{0.6}n^{0.6} \\
\text{(by (7.15))} \quad &\geqslant \frac{\eta^2}{10^4}\big(e_{G_\nabla}(\mathbb{XA}, \mathbb{XA} \cup \mathbb{XB}) - 2e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) - 2|\mathtt{P}|\Omega^* k\big) - k^{0.6}n^{0.6} \\
\text{(by (K1), (7.94), (7.95))} \quad &\geqslant \frac{\eta^2}{10^4}\Big(\frac{\eta kn}{4} - \frac{2\eta^{13}kn}{10^{28}(\Omega^*)^3} - \frac{4\eta^{10}kn}{10^{21}\Omega^*}\Big) - k^{0.6}n^{0.6} \\
&> \frac{\eta^3 kn}{10^5}\ .
\end{aligned} \tag{7.96}$$

We consider the following two complementary cases:

(cA) $e_{G_\nabla}((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}) \geqslant 40\rho kn$.

(cB) $e_{G_\nabla}((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}) < 40\rho kn$.

Note that $\mathbb{XA} \setminus \mathtt{P} \subseteq \mathbb{YA}$, and $(\mathbb{XA} \cup \mathbb{XB}) \setminus \mathtt{P} \subseteq \mathbb{YB}$. We shall now define in each of the cases (**cA**) and (**cB**) certain sets $\mathbb{YA}_1, \mathbb{YA}_2$ which will have a minimum number of edges between them. Although the definition of these sets is different for the cases (**cA**) and (**cB**), for ease of notation they receive the same names.

In case (**cA**) a standard argument (take a maximal cut) gives disjoint sets $\mathbb{YA}_1, \mathbb{YA}_2 \subseteq (\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YA}$ with

$$\begin{aligned} e_{G_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) &\geqslant \frac{1}{2}(e_{G_\nabla}(\mathbb{XA} \setminus \mathtt{P})^{\restriction 0} - |\bar{V} \cup \mathbb{F}| \cdot \Omega^* k) \\ \text{(by Def 7.6(1) and by (7.17))} \quad &\geqslant \frac{1}{2}(40\rho kn - 2\varepsilon\Omega^* kn) \\ &> 19\rho kn\ . \end{aligned} \tag{7.97}$$

Let us now define $\mathbb{YA}_1, \mathbb{YA}_2$ for case (**cB**). Property 6 of Setting 7.4 implies that

$$|\mathtt{P}_2| \leqslant \sqrt{\gamma} n\ . \tag{7.98}$$

Also, by Definition 7.6(7) we have

$$\begin{aligned} e_{G_\nabla}(\mathbb{XA}) &\leqslant \frac{1}{\mathfrak{p}_0^2}\Big(e_{G_\nabla}((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}) + k^{0.6} n^{0.6}\Big) + e_{G_\nabla}(\boldsymbol{\Psi}, \mathbb{XA}) + |\mathtt{P}|\Omega^* k \\ \text{(by (7.15), (\textbf{cB}), (7.94), and (7.95))} \quad &\leqslant \frac{10^4}{\eta^2} \cdot \Big(40\rho kn + k^{0.6}n^{0.6}\Big) + \frac{\eta^{13}}{10^{28}(\Omega^*)^3} kn + \frac{\eta^{10}}{10^{20}\Omega^*} kn \\ \text{(by (7.3))} \quad &< \frac{\eta^8}{10^{15}\Omega^*} kn\ . \end{aligned}$$

Consequently,

$$|\mathtt{P}_3| \cdot \frac{\eta^3 k}{10^3} \leqslant e_{G_\nabla}(\mathtt{P}_3, \mathbb{XA}) \leqslant 2 \cdot \frac{\eta^8}{10^{15}\Omega^*} kn,$$

and thus,

$$|\mathtt{P}_3| \leqslant 2 \cdot \frac{\eta^5}{10^{12}\Omega^*} n\ . \tag{7.99}$$

Set $\mathbb{YA}_1 := (\mathbb{XA} \setminus (\mathtt{P} \cup \mathtt{P}_2 \cup \mathtt{P}_3 \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YA}$ and $\mathbb{YA}_2 := (\mathbb{XB} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YB}$. Then the sets $\mathbb{YA}_1$ and $\mathbb{YA}_2$ are disjoint and we have

$$\begin{aligned} e_{G_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) &\geqslant e_{G_\nabla}\Big((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}, ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathtt{P})^{\restriction 0}\Big) - 2e_{G_\nabla}((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}) \\ &\quad - (|\mathtt{P}_2| + |\mathtt{P}_3| + 2|\bar{V}| + 2|\mathbb{F}|) \cdot \Omega^* k \\ \text{(by (7.96), (\textbf{cB}), (7.98), (7.99), D7.6(1), (7.17))} \quad &\geqslant \frac{\eta^3 kn}{10^5} - 80\rho kn - \sqrt{\gamma}\Omega^* kn - \frac{2\eta^5}{10^{12}} kn - 4\varepsilon\Omega^* kn \\ &\overset{(7.3)}{\geqslant} 19\rho kn\ . \end{aligned} \tag{7.100}$$

We have thus defined $\mathbb{YA}_1, \mathbb{YA}_2$ for both cases (**cA**) and (**cB**).

Observe first that if $e_{G_{\exp}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho kn$ then we may apply Lemma 7.34 to obtain Configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$. Hence, from now on, let us assume that $e_{G_{\exp}}(\mathbb{YA}_1, \mathbb{YA}_2) > 2\rho kn$. Then by (7.97) and (7.100) we have that

$$e_{G_{\mathcal{D}}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 17\rho kn.$$

We fix a family $\mathcal{D}_\nabla$ as in Lemma 7.5. In particular, we have

$$e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 16\rho k n. \tag{7.101}$$

Let $R := \mathbf{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right)$. For $i = 1, 2$ define

$$\begin{aligned} \mathbb{Y}_i^{(1)} &:= \mathbf{shadow}_G(V(G_{\exp}), \rho k) \cap \mathbb{YA}_i\ , \\ \mathbb{Y}_i^{(2)} &:= (V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{YA}_i) \setminus \mathbb{Y}_i^{(1)}\ , \\ \mathbb{Y}_i^{(3)} &:= (R \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)})\ , \\ \mathbb{Y}_i^{(4)} &:= (\mathfrak{A} \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)} \cup \mathbb{Y}_i^{(3)})\ , \\ \mathbb{Y}_i^{(5)} &:= \mathbb{YA}_i \setminus (\mathbb{Y}_i^{(1)} \cup \ldots \cup \mathbb{Y}_i^{(4)})\ . \end{aligned} \tag{7.102}$$

Clearly, the sets $\mathbb{Y}_i^{(j)}$ partition $\mathbb{YA}_i$ for $i = 1, 2$.

We now present two lemmas (one for case **(cA)** and one for case **(cB)**) which help to distinguish several subcases based on the majority type of edges we find between $\mathbb{YA}_1$ and $\mathbb{YA}_2$. The first of the two lemmas follows by simple counting from (7.101).

Lemma 7.36. *In case* **(cB)***, we have one of the following.*

(t1) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2\right) \geqslant 2\rho k n$,

(t2) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2\right) \geqslant 2\rho k n$,

(t3) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2\right) \geqslant 2\rho k n$,

(t4) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2\right) \geqslant 2\rho k n$, *or*

(t5) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(5)}, \mathbb{YA}_2\right) \geqslant 2\rho k n$.

Our second lemma is a bit more involved.

Lemma 7.37. *In case* **(cA)***, we have one of the following.*

(t1) $e_{\mathcal{D}_\nabla}(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2) + e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{Y}_2^{(1)}) \geqslant 4\rho k n$,

(t2) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2 \setminus \mathbb{Y}_2^{(1)}\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus \mathbb{Y}_1^{(1)}, \mathbb{Y}_2^{(2)}\right) \geqslant 4\rho k n$,

(t3) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)})\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)}), \mathbb{Y}_2^{(3)}\right) \geqslant 4\rho k n$, *or*

(t5) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(5)}, \mathbb{Y}_2^{(5)}\right) \geqslant 2\rho k n$.

Proof. By (7.101), we only need to establish that

$$e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)} \cup \mathbb{Y}_1^{(3)}), \mathbb{Y}_2^{(4)}\right) < \rho k n\ .$$

For this, note that $\mathbb{Y}_1^{(4)} \subseteq \mathfrak{A}$ and that $\mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})$ is disjoint from $V_{\leadsto\mathfrak{A}}$. Thus we have $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})\right) < \frac{\rho k n}{100\Omega^*}$. We can bound the other summand using a symmetric argument. □

Next, we prove a lemma that will provide the crucial step for finishing case **(K1)**.

Lemma 7.38. *Let G^* be the spanning subgraph of $G_{\mathcal{D}}$ formed by the edges of $\mathcal{D}_\nabla$. If there are two disjoint sets Z_1 and Z_2 with $e_{G^*}(Z_1, Z_2) \geqslant 2\rho k n$ then there exists an $(\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\widehat{\alpha}\rho k}{\Omega^*})$-semiregular matching $\mathcal{N}$ in G^* with $V_i(\mathcal{N}) \subseteq Z_i$ $(i = 1, 2)$, and $|V(\mathcal{N})| \geqslant \frac{\rho n}{\Omega^*}$.*

Proof. As the maximum degree G^* is bounded by $\Omega^* k$, we have $|Z_1| \geqslant \frac{2\rho n}{\Omega^*} \geqslant \frac{2\rho k}{\Omega^*}$. Thus,

$$(G^*, \mathcal{D}_\nabla, G^*[Z_1, Z_2], \{Z_1\}) \in \mathcal{G}\left(v(G_{\mathcal{D}}), k, \Omega^*, \frac{\gamma^3}{4}, \frac{\rho}{\Omega^*}, 2\rho\right)\ .$$

Lemma 5.6 (which applies with these parameters by the choice of $\widehat{\alpha}$ and k_0 by (7.3), also cf. page 154 for the precise choice) immediately gives the desired output. □

We use Lemma 7.38 with Z_1, Z_2 being the pair of sets containing many edges as in the cases **(t1)**–**(t3)** and **(t5)** of Lemma 7.37[21] and **(t1)**–**(t5)** of Lemma 7.36. The lemma outputs a semiregular matching $\mathcal{M}_{\rhd\text{L}7.35} := \mathcal{N}_{\rhd\text{L}7.38}$. This matching is a basis of the input for Lemma 7.35**(M2)** (subcase **(t1)**–**(t3)**, **(t5)**, or **(t3–5)**). Thus, we get one of the configurations ($\diamond$**6**)–($\diamond$**10**) as in the statement of the lemma. This finishes the proof for case **(K1)**.

Let us now turn our attention to case **(K2)**. For every pair $(X, Y) \in \mathcal{M}_{\text{good}}$, let $X' \subseteq X^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$ and $Y' \subseteq Y^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$ be maximal with $|X'| = |Y'|$. Define $\mathcal{N} := \{(X', Y') : (X, Y) \in \mathcal{M}_{\text{good}}\ ,\ |X'| \geqslant \frac{\eta^2\mathfrak{c}}{2\cdot 10^3}\}$. By Lemma 7.8, and using (7.3) and (7.15), we know that

$$|V(\mathcal{M}_{\text{good}}^{\restriction 0})| \geqslant \frac{\eta^2 n}{400}.$$

Therefore, we have

$$\begin{aligned} |V(\mathcal{N})| &\geqslant |V(\mathcal{M}_{\text{good}}^{\restriction 0})| - 2|\mathtt{P} \cup \bar{V} \cup \mathbb{F}| - 2\frac{\eta^2 n}{2 \cdot 10^3} \\ \text{(by \textbf{(K2)}, (7.95), Def7.6(1), (7.17))} \quad &\geqslant \frac{\eta^2 n}{400} - \frac{4 \cdot \eta^{10} n}{10^{21}(\Omega^*)^2} - 4\varepsilon n - \frac{\eta^2 n}{10^3} \\ &> \frac{\eta^2 n}{1000}\ . \end{aligned} \tag{7.103}$$

[21] The quantities in Lemma 7.37 have two summands. We take the sets Z_1,Z_2 as those appearing in the majority summand.

By Fact 2.7, $\mathcal{N}$ is a $(\frac{4\cdot 10^3\varepsilon'}{\eta^2}, \frac{\gamma^2}{2}, \frac{\eta^2\mathfrak{c}}{2\cdot 10^3})$-semiregular matching.

We use the definitions of the sets $\mathbb{Y}_i^{(1)}, \ldots, \mathbb{Y}_i^{(5)}$ as given in (7.102) with $\mathbb{YA}_i := V_i(\mathcal{N})$ $(i = 1, 2)$. As $V(\mathcal{N}) \subseteq V(G_{\mathrm{reg}})$, we have that $\mathbb{Y}_i^{(4)} = \emptyset$ $(i = 1, 2)$. A set $X \in \mathcal{V}_i(\mathcal{N})$ is said to be of *Type 1* if $\left|X \cap \mathbb{Y}_i^{(1)}\right| \geqslant \frac{1}{4}|X|$. Analogously, we define elements of $\mathcal{V}(\mathcal{N})$ of *Type 2*, *Type 3*, and *Type 5*.

By (7.103) and as $V(\mathcal{M}_{\mathrm{good}}) \subseteq \mathbb{XA}$, we are in subcase (**cA**). For each $(X_1, X_2) \in \mathcal{N}$ with at least one $X_i \in \{X_1, X_2\}$ being of Type 1, set $X_i' := X_i \cap \mathbb{Y}_i^{(1)}$ and take an arbitrary set $X_{3-i}' \subseteq X_{3-i}$ of size $|X_i'|$. Note that by Fact 2.7 (X_i', X_{3-i}') forms a $\frac{10^5\varepsilon'}{\eta^2}$-regular pair of density at least $\gamma^2/4$. We let $\mathcal{N}_1$ be the semiregular matching consisting of all pairs (X_i', X_{3-i}') obtained in this way.[22]

Likewise, we construct $\mathcal{N}_2, \mathcal{N}_3$ and $\mathcal{N}_5$ using the features of Type 2, 3, and 5. Observe that the matchings $\mathcal{N}_i$ may intersect.

Because of (7.103) and since we included at least one quarter of each $\mathcal{N}$-edge into one of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ and $\mathcal{N}_5$, one of the semiregular matchings $\mathcal{N}_i$ satisfies $|V(\mathcal{N}_i)| \geqslant \frac{\eta^2 n}{16\cdot 1000} \geqslant \frac{\rho}{\Omega^*}n$. So, $\mathcal{N}_i$ serves as a matching $\mathcal{M}_{\triangleright \mathrm{L7.35}}$ for Lemma 7.35**(M1)**. Thus, we get one of the configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{10})$ as in the statement of the lemma. This finishes case **(K2)**.

7.7.3 Proof of Lemma 7.33

Set $\tilde{\eta} := \frac{\eta^{13}}{10^{28}(\Omega^*)^3}$. Define $\mathrm{N}^{\uparrow} := \{v \in V(G) : \deg_{G_\nabla}(v, \mathbf{\Psi}) \geqslant k\}$, and $\mathrm{N}^{\downarrow} := \mathrm{N}_{G_\nabla}(\mathbf{\Psi}) \setminus \mathrm{N}^{\uparrow}$. Recall that by the definition of the class $\mathbf{LKSsmall}(n, k, \eta)$, the set $\mathbf{\Psi}$ is independent, and thus the sets $\mathrm{N}^{\uparrow}$ and $\mathrm{N}^{\downarrow}$ are disjoint from $\mathbf{\Psi}$. Also, using the same definition, we have

$$\mathrm{N}_{G_\nabla}(\mathbf{\Psi}) \subseteq \mathbb{L}_{\eta,k}(G) \setminus \mathbf{\Psi} \quad \text{, and thus} \tag{7.104}$$

$$e_{G_\nabla}(\mathbf{\Psi}, B) = e_{G_\nabla}(\mathbf{\Psi}, B \cap \mathbb{L}_{\eta,k}(G)) \text{ for any } B \subseteq V(G). \tag{7.105}$$

We shall distinguish two cases.

Case A: $e_{G_\nabla}(\mathbf{\Psi}, \mathrm{N}^{\uparrow}) \geqslant e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB})/8$.

Let us focus on the bipartite subgraph H' of G_∇ induced by the sets $\mathbf{\Psi}$ and $\mathrm{N}^{\uparrow}$. Obviously, the average degree of the vertices of $\mathrm{N}^{\uparrow}$ in H' is at least k.

First, suppose that $|\mathbf{\Psi}| \leqslant |\mathrm{N}^{\uparrow}|$. Then, the average degree of $\mathbf{\Psi}$ in H' is at least k, and hence, the average degree of H' is at least k. Thus, there exists a bipartite subgraph $H \subseteq H'$ with $\deg^{\min}(H) \geqslant k/2$. Furthermore, $\deg^{\min}{}_{G_\nabla}(V(H)) \geqslant k$. We conclude that we are in Configuration $(\diamond\mathbf{1})$.

Now, suppose $|\mathbf{\Psi}| > |\mathrm{N}^{\uparrow}|$. Using the bounds given by Case A, and using (7.90), we get

$$|\mathrm{N}^{\uparrow}| \geqslant \frac{e_{G_\nabla}(\mathbf{\Psi}, \mathrm{N}^{\uparrow})}{\Omega^* k} \geqslant \frac{\tilde{\eta} k n}{8\Omega^* k} = \frac{\tilde{\eta} n}{8\Omega^*} \ .$$

Therefore, we have

$$e(G) \geqslant \sum_{v \in \mathbf{\Psi}} \deg_{G_\nabla}(v) \geqslant |\mathbf{\Psi}|\Omega^{**}k > |\mathrm{N}^{\uparrow}|\Omega^{**}k \geqslant \frac{\tilde{\eta} n}{8\Omega^*}\Omega^{**}k \overset{(7.3)}{\geqslant} kn \ ,$$

a contradiction to Property 3 of Definition 2.6.

[22] Note that we are thus changing the orientation of some subpairs.

Case B: $e_{G_\nabla}(\mathbf{\Psi}, \mathrm{N}^\uparrow) < e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB})/8$.
Consequently, we get

$$e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^\uparrow) \geqslant \frac{7}{8} e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \overset{(7.90)}{\geqslant} \frac{7}{8}\tilde{\eta}kn \,. \tag{7.106}$$

We now apply Lemma 7.27 to G_∇ with input sets $P_{\triangleright \mathrm{L}7.27} := \mathbf{\Psi}$, $Q_{\triangleright \mathrm{L}7.27} := \mathbb{L}_{\eta,k}(G) \setminus \mathbf{\Psi}$, $Y_{\triangleright \mathrm{L}7.27} := \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla)$, and parameters $\psi_{\triangleright \mathrm{L}7.27} := \tilde{\eta}/100$, $\Gamma_{\triangleright \mathrm{L}7.27} := \Omega^*$, and $\Omega_{\triangleright \mathrm{L}7.27} := \Omega^{**}$. Assumption (7.65) of the lemma follows from (7.104). The lemma yields three sets $L'' := Q''_{\triangleright \mathrm{L}7.27}$, $L' := Q'_{\triangleright \mathrm{L}7.27}$, $\mathbf{\Psi}' := P'_{\triangleright \mathrm{L}7.27}$, and it is easy to check that these witness Preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^6(\Omega^*)^2})$.

Recall that $e(G) \leqslant kn$. Since by the definition of $Y_{\triangleright \mathrm{L}7.27}$, we have $|Y_{\triangleright \mathrm{L}7.27}| \leqslant \frac{40\rho}{\eta}n$, we obtain from Lemma 7.27(d) that

$$\begin{aligned} e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbf{\Psi}', L'') &\leqslant \frac{\tilde{\eta}}{100} e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) + \frac{|Y_{\triangleright \mathrm{L}7.27}|200(\Omega^*)^2k}{\tilde{\eta}} \\ &\leqslant \frac{\tilde{\eta}}{100}kn + \frac{40\rho n}{\eta} \cdot \frac{200(\Omega^*)^2 k}{\tilde{\eta}} \\ &\overset{(7.3)}{\leqslant} \frac{\tilde{\eta}}{2}kn. \end{aligned} \tag{7.107}$$

So,

$$\begin{aligned} e_{G_\nabla}\big(\mathbf{\Psi}', (L'' \cap (\mathbb{XA} \cup \mathbb{XB})) \setminus \mathrm{N}^\uparrow\big) &\geqslant e_{G_\nabla}\big(\mathbf{\Psi}, (\mathbb{L}_{\eta,k}(G) \cap (\mathbb{XA} \cup \mathbb{XB})) \setminus \mathrm{N}^\uparrow\big) \\ &\quad - \big(e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)\big) - e_{G_\nabla}(\mathbf{\Psi}', L'')\big) \\ &= e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^\uparrow) \\ &\quad - \big(e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)\big) - e_{G_\nabla}(\mathbf{\Psi}', L'')\big) \\ &\overset{(7.107)}{\geqslant} e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^\uparrow) - \frac{\tilde{\eta}}{2}kn \\ &\overset{(7.106)}{\geqslant} \frac{3}{8}\tilde{\eta}kn \,. \end{aligned} \tag{7.108}$$

We define

$$\mathbf{\Psi}^* := \left\{ v \in \mathbf{\Psi}' \,:\, \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^\downarrow) \geqslant \sqrt{\Omega^{**}}k \right\} \,.$$

Using that $e(G) \leqslant kn$, we shall show the following.

Lemma 7.39. *We have* $e_{G_\nabla}(\mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^\downarrow) \geqslant \frac{1}{8}\tilde{\eta}kn$.

Proof. Suppose otherwise. Then by (7.108), we obtain that

$$e_{G_\nabla}(\mathbf{\Psi}' \setminus \mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^\downarrow) \geqslant \frac{1}{4}\tilde{\eta}kn \,.$$

On the other hand, by the definition of $\mathbf{\Psi}^*$,

$$|\mathbf{\Psi}' \setminus \mathbf{\Psi}^*|\sqrt{\Omega^{**}}k \geqslant e_{G_\nabla}(\mathbf{\Psi}' \setminus \mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^\downarrow) \,.$$

Consequently, we have

$$|\mathbf{\Psi}' \setminus \mathbf{\Psi}^*| \geqslant \frac{\tilde{\eta}kn}{4\sqrt{\Omega^{**}}k} = \frac{\tilde{\eta}n}{4\sqrt{\Omega^{**}}} .$$

Thus, as $\mathbf{\Psi}$ is independent,

$$e(G) \geqslant \sum_{v \in \mathbf{\Psi}} \deg_{G_\nabla}(v) \geqslant |\mathbf{\Psi}|\Omega^{**}k \geqslant |\mathbf{\Psi}' \setminus \mathbf{\Psi}^*|\Omega^{**}k \geqslant \frac{\tilde{\eta}}{4}\sqrt{\Omega^{**}}kn \overset{(7.3)}{>} kn ,$$

a contradiction. □

Let us define $O := \mathbf{shadow}_{G_\nabla}(\mathfrak{A}, \gamma k)$. Next, we define

$$\begin{aligned} N_1 &:= V(G_{\mathrm{exp}}) \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} , \\ N_2 &:= \mathfrak{A} \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} , \\ N_3 &:= O \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} , \text{ and} \\ N_4 &:= (L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \setminus (N_1 \cup N_2 \cup N_3) . \end{aligned}$$

Observe that

$$O \cap N_4 = \emptyset . \tag{7.109}$$

Further, for $i = 1, \ldots, 4$ define

$$C_i := \left\{ v \in \mathbf{\Psi}^* \ : \ \deg_{G_\nabla}(v, N_i) \geqslant \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow})/4 \right\} .$$

Easy counting gives that there exists an index $i \in [4]$ such that

$$e_{G_\nabla}(C_i, N_i) \geqslant \frac{1}{16} e_{G_\nabla}(\mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \overset{\mathrm{L}7.39}{\geqslant} \frac{1}{128}\tilde{\eta}kn . \tag{7.110}$$

Set $Y := (\mathbb{XA} \cup \mathbb{XB}) \setminus (\mathbb{YB} \cup \mathbf{\Psi}) = (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}$, and $\eta_{\triangleright \mathrm{L}7.28} = \eta_{\triangleright \mathrm{L}7.29} := \frac{1}{128}\tilde{\eta}$. By Lemma 7.9 we have

$$|Y| < \frac{\eta_{\triangleright \mathrm{L}7.28} n}{4\Omega^*} . \tag{7.111}$$

We split the rest of the proof into four subcases according to the value of i.

Subcase B, $i = 1$.

We shall apply Lemma 7.28 with $r_{\triangleright \mathrm{L}7.28} := 2$, $\Omega^*_{\triangleright \mathrm{L}7.28} := \Omega^*, \Omega^{**}_{\triangleright \mathrm{L}7.28} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright \mathrm{L}7.28} := \frac{\eta_{\triangleright \mathrm{L}7.28}\rho^2}{100(\Omega^*)^2}$, $\gamma_{\triangleright \mathrm{L}7.28} := \rho$, $\eta_{\triangleright \mathrm{L}7.28}$, $X_0 := C_1$, $X_1 := N_1$, and $X_2 := V(G_{\mathrm{exp}})$, and Y, and the graph $G_{\triangleright \mathrm{L}7.28}$, which is formed by the vertices of G, with all edges from $E(G_\nabla)$ that are in $E(G_{\mathrm{exp}})$ or that are incident with $\mathbf{\Psi}$. We briefly verify the assumptions of Lemma 7.28. First of all the choice of $\delta_{\triangleright \mathrm{L}7.28}$ guarantees that $\left(\frac{3\Omega^*_{\triangleright \mathrm{L}7.28}}{\gamma_{\triangleright \mathrm{L}7.28}}\right)^2 \delta_{\triangleright \mathrm{L}7.28} < \frac{\eta_{\triangleright \mathrm{L}7.28}}{10}$. Assumption 1 is given by (7.111). Assumption 2 holds since we assume that (7.110) is satisfied for $i = 1$ and by definition of $\eta_{\triangleright \mathrm{L}7.28}$. Assumption 3 follows from the definitions of C_1 and of $\mathbf{\Psi}^*$. Assumption 4 follows from the fact that $X_1 \subseteq V(G_{\mathrm{exp}}) = X_2$, and since $\deg^{\min}(G_{\mathrm{exp}}) > \rho k$ which is guaranteed by the definition of

a $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon',\nu,\rho)$-sparse decomposition. This definition also guarantees Assumption 5, as $Y \cup X_1 \cup X_2 \subseteq V(G) \setminus \mathbf{\Psi}$.

Lemma 7.28 outputs sets $\mathbf{\Psi}'' := X_0'$, $V_1 := X_1'$, $V_2 := X_2'$ with $\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \sqrt[4]{\Omega^{**}}k/2$ (by (d)), $\mathrm{deg^{max}}_{G_{\exp}}(V_1, X_2 \setminus V_2) < \rho k/2$ (by (c)), $\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta_{\triangleright L7.28} k$ (by (b)), and $\mathrm{deg^{min}}_{G_{\exp}}(V_2, V_1) \geqslant \delta_{\triangleright L7.28} k$ (by (b)). By (a), we have that $V_1 \subseteq \mathbb{YB} \cap L''$. As $\mathrm{deg^{min}}_{G_{\exp}}(V_1, X_2) \geqslant \mathrm{deg^{min}}(G_{\exp}) \geqslant \rho k$, we have $\mathrm{deg^{min}}_{G_{\exp}}(V_1, V_2) \geqslant \mathrm{deg^{min}}_{G_{\exp}}(V_1, X_2) - \mathrm{deg^{max}}_{G_{\exp}}(V_1, X_2 \setminus V_2) \geqslant \delta_{\triangleright L7.28} k$.

Since L', L'' and $\mathbf{\Psi}'$ witness Preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}})$, this verifies that we have Configuration $(\diamond\mathbf{2})\left(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde{\eta}\rho^2}{12800(\Omega^*)^2}\right)$.

Subcase B, $i = 2$.

We apply Lemma 7.28 with numerical parameters $r_{\triangleright L7.28} := 2$, $\Omega^*_{\triangleright L7.28} := \Omega^*$, $\Omega^{**}_{\triangleright L7.28} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright L7.28} := \frac{\eta_{\triangleright L7.28}\gamma^2}{100(\Omega^*)^2}$, $\gamma_{\triangleright L7.28} := \gamma$, and $\eta_{\triangleright L7.28}$. Further input to the lemma are sets $X_0 := C_2$, $X_1 := N_2$, and $X_2 := V(G) \setminus \mathbf{\Psi}$, and the set Y. The underlying graph $G_{\triangleright L7.28}$ is the graph $G_{\mathcal{D}}$ with all egdes incident with $\mathbf{\Psi}$ added. Verifying assumptions of Lemma 7.28 is analogous to Subcase B, $i = 1$ with the exception of Assumption 4. Let us therefore turn to verify it. To this end, it suffices to observe that each each vertex in X_1 is contained in at least one $(\gamma k, \gamma)$-dense spot from $\mathcal{D}$ (cf. Definition 4.6), and thus has degree at least γk in X_2.

The output of Lemma 7.28 are sets X_0', X_1', and X_2' which witness Configuration $(\diamond\mathbf{3})(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde{\eta}\gamma^2}{12800(\Omega^*)^2})$. In fact, the only thing not analogous to the preceding subcase is that we have to check (7.31), in other words, we have to verify that

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}\big(X_1', V(G) \setminus (X_2' \cup \mathbf{\Psi})\big) \leqslant \frac{\gamma k}{2} \,.$$

As $V(G) \setminus (X_2' \cup \mathbf{\Psi}) = X_2 \setminus X_2'$, this follows from (c) of Lemma 7.28.

Subcase B, $i = 3$.

We apply Lemma 7.28 with numerical parameters $r_{\triangleright L7.28} := 3$, $\Omega^*_{\triangleright L7.28} := \Omega^*$, $\Omega^{**}_{\triangleright L7.28} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright L7.28} := \frac{\eta_{\triangleright L7.28}\gamma^3}{300(\Omega^*)^3}$, $\gamma_{\triangleright L7.28} := \gamma$, and $\eta_{\triangleright L7.28}$. Further inputs are the sets $X_0 := C_3$, $X_1 := N_3$, $X_2 := \mathfrak{A}$, and $X_3 := V(G) \setminus \mathbf{\Psi}$, and the set Y. The underlying graph is $G_{\triangleright L7.28} := G_\nabla \cup G_{\mathcal{D}}$. Verifying assumptions Lemma 7.28 is analogous to Subcase B, $i = 1$, only for Assumption 4 we observe that $\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(X_1, X_2) \geqslant \mathrm{deg^{min}}_{G_\nabla}(X_1, X_2) \geqslant \gamma k$ by definition of $X_1 = N_3 \subseteq O$, and $\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(X_2, X_3) \geqslant \mathrm{deg^{min}}_{G_{\mathcal{D}}}(X_2, X_3) \geqslant \gamma k$ for the same reason as in Subcase B, $i = 2$.

Lemma 7.28 outputs Configuration $(\diamond\mathbf{4})\left(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde{\eta}\gamma^3}{38400(\Omega^*)^3}\right)$, with $\mathbf{\Psi}'' := X_0'$, $V_1 := X_1'$, $\mathfrak{A}' := X_2'$ and $V_2 := X_3'$. Indeed, all calculations are similar to the ones in the preceding two subcases, we only need to note additionally that $\mathrm{deg^{min}}_{G_\nabla \cup G_{\mathcal{D}}}(V_1, \mathfrak{A}') \geqslant \frac{\gamma k}{2} \frac{\tilde{\eta}\gamma^3 k}{38400(\Omega^*)^3}$, which follows from the definition of N_3 and of O.

Subcase B, $i = 4$.

We have $\mathbf{V} \neq \emptyset$ and $\mathfrak{c}$ is the size of an arbitrary cluster in $\mathbf{V}$. We are going apply Lemma 7.29 with $\delta_{\triangleright L7.29} := \eta_{\triangleright L7.29}/100$, $\eta_{\triangleright L7.29}$, $h_{\triangleright L7.29} := \eta_{\triangleright L7.29}\mathfrak{c}/(100\Omega^*)$, $\Omega^*_{\triangleright L7.29} := \Omega^*$, $\Omega^{**}_{\triangleright L7.29} := \sqrt{\Omega^{**}}/4$ and sets $X_0 := C_4$, $X_1 := N_4$, and Y. The underlying graph is $G_{\triangleright L7.29} := G_\nabla$, and $\mathcal{C}_{\triangleright L7.29}$ is the

set of clusters $\mathbf{V}$.

The fact $e(G) \leqslant kn$ together with (7.110) and the choice of $\eta_{\triangleright\text{L7.29}}$ gives Assumption 2 of Lemma 7.29. The choice of C_4 and $\mathbf{\Psi}^*$ gives Assumption 3. The fact that $X_1 \cap \mathbf{\Psi} = \emptyset$ yields Assumption 4. With the help of (7.3) it is easy to check Assumption 1. Inequality (7.111) implies Assumption 5. To verify Assumption 6, it is enough to use that $|\mathcal{C}_{\triangleright\text{L7.29}}| \leqslant \frac{n}{\mathfrak{c}}$. We have thus verified all the assumptions of Lemma 7.29.

We claim that Lemma 7.29 outputs Configuration $(\diamond\mathbf{5})\left(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde{\eta}}{12800}, \frac{\eta}{2}, \frac{\tilde{\eta}}{12800\Omega^*}\right)$, with $\mathbf{\Psi}'' := X_0'$ and $V_1 := X_1'$. In fact, all conditions of the configuration, except condition (7.39), which we check below, are easy to verify. (Note that $V_1 \subseteq \mathbb{YB}$ since $V_1 \subseteq X_1 = N_4 \subseteq \mathbb{XA} \cup \mathbb{XB}$. Also, $V_1 \subseteq L''$, and thus disjoint from $\mathbf{\Psi}$. Moreover, by the conditions of Lemma 7.29, V_1 is disjoint from Y. So, $V_1 \subseteq \mathbb{YB}$.) For (7.39), observe that (7.109) implies that $\deg^{\max}{}_{G_\nabla}(N_4, \mathfrak{A}) \leqslant \gamma k$. Further, we have $X_1' \subseteq N_4 \setminus Y$. So for all $x \in X_1' \subseteq \mathrm{N}^{\downarrow} \setminus Y$, we have that $\deg_{G_\nabla}(x, V(G) \setminus \mathbf{\Psi}) \geqslant \frac{9\eta k}{10}$. As $N_4 \subseteq \bigcup \mathbf{V} \setminus V(G_{\exp})$, we obtain $\deg_{G_{\text{reg}}}(x) \geqslant \frac{9\eta k}{10} - \gamma k \geqslant \frac{\eta k}{2}$, fulfilling (7.39).

7.7.4 Proof of Lemma 7.34

Set $\mathbb{YA}_1' := \{v \in \mathbb{YA}_1 : \deg_{G_{\exp}}(v, \mathbb{YA}_2) \geqslant \rho k\}$. By (7.91) we have

$$e_{G_{\exp}}(\mathbb{YA}_1', \mathbb{YA}_2) \geqslant \rho kn \ . \tag{7.112}$$

Set $r_{\triangleright\text{L7.30}} := 3$, $\Omega_{\triangleright\text{L7.30}} := \Omega^*$, $\gamma_{\triangleright\text{L7.30}} := \frac{\rho\eta}{10^3}$, $\delta_{\triangleright\text{L7.30}} := \frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}$, $\eta_{\triangleright\text{L7.30}} := \rho$. Observe that (7.76) is satisfied for these parameters. Set $Y_{\triangleright\text{L7.30}} := \bar{V}$, $X_0 := \mathbb{YA}_2$, $X_1 := \mathbb{YA}_1'$, $X_2 = X_3 := V(G_{\exp})^{\restriction 1}$, and $V := V(G)$. Let $E_2 := E(G_\nabla)$, and $E_1 = E_3 := E(G_{\exp})$. We now briefly verify conditions 1–4 of Lemma 7.30. Condition 1 follows from Definition 7.6(1) and (7.3). Condition 2 follows from (7.112). Using Definition 7.6(6), (7.15) and (7.3), we see that Condition 3 for $i = 1$ follows from the definition of $\mathbb{YA}_1'$, and for $i = 2$ from the fact that $\deg^{\min}(G_{\exp}) \geqslant \rho k$. Last, Condition 4 follows from the fact that $\bigcup_{i=0}^3 X_i$ is disjoint from $\mathbf{\Psi}$.

Lemma 7.30 yields four non-empty sets $X_0', \ldots, X_3'$. By assertions (a), (b), (c), and hypothesis 3 of Lemma 7.30, for all $i \in \{0,1,2,3\}$, $j \in \{i-1, i+1\} \setminus \{-1, 4\}$ we have

$$\deg^{\min}{}_{H_{i,j}}(X_i', X_j') \geqslant \delta_{\triangleright\text{L7.30}} k, \tag{7.113}$$

where $H_{i,j} = G_{\exp}$, except for $\{i,j\} = \{1,2\}$, where $H_{i,j} = G_\nabla$.

Thus, the sets X_0' and X_1' witness Preconfiguration $(\mathbf{exp})(\delta_{\triangleright\text{L7.30}})$. By Lemma 7.10, and by (7.92) and (7.93), the pair X_0', X_1' together with the cover $\mathcal{F}$ from (7.13) witnesses either Preconfiguration $(\heartsuit\mathbf{1})(\frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$ (with respect to $\mathcal{F}$) or Preconfiguration $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Notice that (7.113) establishes the properties (7.48)–(7.51). Thus the sets $X_0', \ldots, X_3'$ witness Configuration $(\diamond\mathbf{6})(\delta_{\triangleright\text{L7.30}}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

7.7.5 Proof of Lemma 7.35

In Lemmas 7.40, 7.41, 7.43, 7.44, 7.45 below, we show that cases (**t1**), (**t2**), (**t3**), **(t3–t5)**, and (**t5**) of Lemma 7.35 lead to configuration $(\diamond\mathbf{6})$, $(\diamond\mathbf{7})$, $(\diamond\mathbf{8})$, $(\diamond\mathbf{9})$, and $(\diamond\mathbf{10})$, respectively. While the

first three of these cases are resolved by a fairly straightforward application of the Cleaning Lemma (Lemma 7.31), the latter two cases require some further non-trivial computations.

Lemma 7.40. *In case* (**t1**) *(of either subcase* (**cA**) *or subcase* (**cB**)*) we obtain Configuration* $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$.

Proof. We use Lemma 7.31 with the following input parameters: $r_{\triangleright\mathrm{L}7.31} := 3$, $\Omega_{\triangleright\mathrm{L}7.31} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.31} := \eta\rho/200$, $\eta_{\triangleright\mathrm{L}7.31} := \rho/(2\Omega^*)$, $\delta_{\triangleright\mathrm{L}7.31} := \eta^3\rho^4/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\triangleright\mathrm{L}7.31} := \bar\varepsilon$, $\mu_{\triangleright\mathrm{L}7.31} := \beta$ and $d_{\triangleright\mathrm{L}7.31} := \bar d$. Note these parameters satisfy the numerical conditions of Lemma 7.31. We use the vertex sets $Y_{\triangleright\mathrm{L}7.31} := \bar V \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 = X_3 := V(G_{\exp})^{\restriction 1}$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 7.31 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_\nabla)$ and $E_3 := E(G_{\exp})$.

Let us verify the conditions of Lemma 7.31. Condition 1 follows from Definition 7.6(1) and (7.17). Condition 2 holds by the assumption on $\mathcal{M}$. Condition 3 follows from Definition 7.6(6) by (7.15), and for $i=1$ also from the definition of $\mathcal{M}$. Conditions 4 hold by the definition of $\mathcal{M}$. Finally, Condition 5 follows from the properties of the sparse decomposition ∇.

The output of Lemma 7.31 are four sets $X_0', \dots, X_3'$. By Lemma 7.10, the sets X_0' and X_1' witness Preconfiguration $(\heartsuit\mathbf{1})(3\eta^3/(2\cdot 10^3), \mathfrak{p}_2(1+\frac{\eta}{20})k)$, or $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1+\frac{\eta}{20})k)$. Further, Lemma 7.31(a) gives that (X_0', X_1') witnesses Preconfiguration $(\mathbf{reg})(4\bar\varepsilon, \bar d/4, \beta/2)$. It is now easy to verify that we have Configuration $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\bar\varepsilon, \frac{\bar d}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$.

This leads to Configuration $(\diamond\mathbf{6})$ with parameters as claimed. Indeed, no matter whether we have **(M1)** or **(M2)**, we have $4\pi \geqslant 4\cdot\frac{10^5\varepsilon'}{\eta^2}$, and $\gamma^3\rho/(32\Omega^*) \leqslant \gamma^2/4$, and $\eta^2\nu/(2\cdot 10^4) \leqslant \eta^2\mathfrak{c}/(8\cdot 10^3 k) \leqslant \eta^2\varepsilon'/(8\cdot 10^3) \leqslant \widehat\alpha\rho/\Omega^*$ (for the latter recall that $\mathfrak{c} \leqslant \varepsilon' k$ by Definition 4.7 (3)). □

Lemma 7.41. *In case* (**t2**) *(of either subcase* (**cA**) *or subcase* (**cB**)*) we obtain Configuration* $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$.

Proof. We use Lemma 7.31 with the following input parameters: $r_{\triangleright\mathrm{L}7.31} := 3$, $\Omega_{\triangleright\mathrm{L}7.31} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.31} := \eta\gamma/200$, $\eta_{\triangleright\mathrm{L}7.31} := \rho/\Omega^*$, $\delta_{\triangleright\mathrm{L}7.31} := \eta^3\gamma^3\rho/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\triangleright\mathrm{L}7.31} := \bar\varepsilon$, $\mu_{\triangleright\mathrm{L}7.31} := \beta$ and $d_{\triangleright\mathrm{L}7.31} := \bar d$. We use the vertex sets $Y_{\triangleright\mathrm{L}7.31} := \bar V \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 := \mathfrak{A}^{\restriction 1}$, $X_3 := \mathfrak{P}_1$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 7.31 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_\nabla)$ and $E_3 := E(G_{\mathcal{D}})$.

The conditions of Lemma 7.31 are verified as before, let us just note that Condition 3 follows from Definition 7.6(6) and by (7.15), and for $i=1$ from the definition of $\mathcal{M}$, while for $i=2$ it holds since $\mathfrak{A}$ is covered by the set $\mathcal{D}$ of $(\gamma k, \gamma)$-dense spots (cf. Definition 4.6).

It is now easy to check that the output of Lemma 7.31 are sets that witness Configuration $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\bar\varepsilon, \frac{\bar d}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$. □

Before proceeding with dealing with cases (**t3**), (**t5**) and **(t3–5)** we state some properties of the matching $\bar{\mathcal{M}} := \big(\mathcal{M}_A \cup \mathcal{M}_B\big)^{\restriction 1}$.

Lemma 7.42. *For* $V_{\mathrm{leftover}} := V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1} \setminus V(\bar{\mathcal{M}})$ *and* $Y_{\bar{\mathcal{M}}} := \bar{V} \cup \mathbb{F} \cup \mathbf{shadow}_{G_{\mathcal{D}}}(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000})$, *we have*

(a) $\bar{\mathcal{M}}$ *is a* $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200})$*-semiregular matching absorbed by* $\mathcal{M}_A \cup \mathcal{M}_B$ *and* $V(\bar{\mathcal{M}}) \subseteq \mathfrak{P}_1$, *and*

(b) $|Y_{\bar{\mathcal{M}}}| \leqslant \frac{3000\varepsilon\Omega^* n}{\eta^2}$.

Proof. Lemma 7.42 (a) follows from Lemma 7.8.

Observe that from properties (1) and (3) of Definition 7.6 we can calculate that

$$|V_{\mathrm{leftover}}| \leqslant 3 \cdot k^{0.9} \cdot |\mathcal{M}_A \cup \mathcal{M}_B| + \left|\bigcup \bar{\mathcal{V}} \cup \bar{\mathcal{V}}^*\right| \leqslant 3 \cdot k^{0.9} \cdot \frac{n}{2\pi\mathfrak{c}} + 2\exp(-k^{0.1}) \overset{(7.3)}{\leqslant} 2\varepsilon n. \tag{7.114}$$

Then

$$\begin{aligned} |Y_{\bar{\mathcal{M}}}| &\leqslant |\bar{V}| + |\mathbb{F}| + \left|\mathbf{shadow}_{G_{\mathcal{D}}}\left(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000}\right)\right| \\ \text{(by Fact 7.1)} \quad &\leqslant |\bar{V}| + |\mathbb{F}| + |V_{\mathrm{leftover}}| \frac{1000\Omega^*}{\eta^2} \\ \text{(by (7.114), D7.6(1), (7.3) (7.17))} \quad &< \frac{3000\varepsilon\Omega^* n}{\eta^2} \,, \end{aligned}$$

as desired for Lemma 7.42(b). □

Lemma 7.43. *In Case* (**t3**)(**cA**) *we obtain Configuration* $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$.

Proof. We use Lemma 7.31 with the following input parameters: $r_{\triangleright\mathrm{L}7.31} := 4$, $\Omega_{\triangleright\mathrm{L}7.31} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.31} := \eta\gamma/200$, $\eta_{\triangleright\mathrm{L}7.31} := \rho/\Omega^*$, $\delta_{\triangleright\mathrm{L}7.31} := \eta^4\gamma^4\rho/(10^{15}(\Omega^*)^5)$, $\varepsilon_{\triangleright\mathrm{L}7.31} := \bar{\varepsilon}$, $\mu_{\triangleright\mathrm{L}7.31} := \beta$ and $d_{\triangleright\mathrm{L}7.31} := \bar{d}$. We use the following vertex sets $Y_{\triangleright\mathrm{L}7.31} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$,

$$X_2 := (\mathbb{L}_{\eta,k}(G) \cap V_{\rightsquigarrow\mathfrak{A}})^{\restriction 0} \setminus \big(V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V_{\not\rightsquigarrow\mathbf{\Psi}} \cup L_{\#} \cup \mathtt{P}_{\mathfrak{A}} \cup \mathtt{P}_1\big) \,,$$

$X_3 := \mathfrak{A}^{\restriction 1}$, $X_4 := \mathfrak{P}_1$, and $V := V(G)$. The partitions $P_i^{(j)}$ of X_0 and X_1 in Lemma 7.31 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 = E_3 := E(G_\nabla)$ and $E_4 := E(G_{\mathcal{D}})$.

Most of the conditions of Lemma 7.31 are verified as before, let us only note the few differences. Condition 1 follows from Lemma 7.42(b). Using Definition 7.6(6) and (7.15), we find that Condition 3 for $i=2$ follows from the definition of $V_{\rightsquigarrow\mathfrak{A}}$, and Condition 3 for $i=3$ holds as it is the same as Condition 3 for $i=2$ in Lemma 7.41. In order to prove Condition 3 for $i=1$ we first observe that since we are in case (**t3**), we have

$$V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right) \setminus (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}}) \,. \tag{7.115}$$

Also, since we in case (**cA**), we have

$$V_1(\mathcal{M}) \cap \mathtt{P} = \emptyset \,. \tag{7.116}$$

Thus, for each $v \in V_1(\mathcal{M})$ we have, using Definition 7.6(6),

$$\begin{aligned}
\deg_{G_\nabla}(v, X_2) &\geqslant \mathfrak{p}_0\Big(\deg_{G_\nabla}(v, (\mathbb{L}_{\eta,k}(G) \cap V_{\rightsquigarrow\mathfrak{A}}) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
&\qquad - \deg_{G_\nabla}(v, V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V_{\not\rightsquigarrow\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_\mathfrak{A} \cup \mathtt{P}_1)\Big) - k^{0.9} \\
\text{(by (7.115) \& (7.116) \& (7.15))} \quad &\geqslant \frac{\eta}{100}\left(\frac{2\eta^2 k}{10^5} - \rho k - \frac{\rho k}{100\Omega^*} - \frac{\eta^2 k}{10^5}\right) - k^{0.9} \\
\text{(by (7.3))} \quad &\geqslant \frac{\eta\gamma k}{200} ,
\end{aligned}$$

which indeed verifies Condition 3 for $i = 1$.

Define $\mathcal{N} := \bar{\mathcal{M}} \setminus \{(X,Y) \in \bar{\mathcal{M}} : X \cup Y \subseteq V(\mathcal{N}_\mathfrak{A})\}$. By Lemma 7.42 (a) we have that $\mathcal{N} \subseteq \bar{\mathcal{M}}$ is a $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200})$-semiregular matching absorbed by $\mathcal{M}_A \cup \mathcal{M}_B$, and that $V(\mathcal{N}) \subseteq \mathfrak{P}_1$.

To see that the output of Lemma 7.31 together with the matching $\mathcal{N}$ leads to Configuration $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$ let us show that (7.62) is satisfied (the other conditions are more easily seen to hold).

For this, let $v \in X_2'$. We have to show that

$$\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) \geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k. \tag{7.117}$$

Note that $v \notin V(G_{\mathrm{exp}})$, and thus $\deg_{G_{\mathrm{exp}}}(v) = 0$. This allows us to calculate as follows:

$$\begin{aligned}
\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \deg_{G_\nabla}(v, \mathfrak{P}_1) - \deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathfrak{A})) - \deg_{G_{\mathrm{reg}}}(v, V_{\mathrm{leftover}}) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) .
\end{aligned} \tag{7.118}$$

We now bound the terms of the right-hand side of (7.118). From Definition 7.6(6) we obtain that $\deg_{G_\nabla}(v, \mathfrak{P}_1) \geqslant \mathfrak{p}_1\left(\deg_{G_\nabla}(v) - \deg_G(v, \mathbf{\Psi})\right) - k^{0.9}$. Lemma 7.31(c) gives that $\deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \leqslant \frac{\eta\gamma k}{400}$. As $v \notin \mathtt{P}_\mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B)$, we have $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathfrak{A})) < \gamma k$. As $v \notin Y_{\bar{\mathcal{M}}}$ and thus $v \notin \mathbf{shadow}_{G_\mathcal{D}}\left(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000}\right)$ we have $\deg_{G_\mathcal{D}}(v, V_{\mathrm{leftover}}) \leqslant \frac{\eta^2 k}{1000}$. Last, recall that $v \notin \mathtt{P}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B)$, and consequently $\deg_{G_{\mathrm{reg}}}(v, V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) < \gamma k$. Putting these bounds together, we find that

$$\begin{aligned}
\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \mathfrak{p}_1\left(\deg_{G_\nabla}(v) - \deg_G(v, \mathbf{\Psi})\right) - \frac{2\eta^2 k}{1000} \\
\text{(as } v \in \mathbb{L}_{\eta,k}(G) \setminus (L_\# \cup V_{\not\rightsquigarrow\mathbf{\Psi}})) \quad &\geqslant \mathfrak{p}_1\left((1+\frac{9\eta}{10})k - \frac{\eta k}{100}\right) - \frac{\eta^2 k}{500} \\
\text{(by (7.15) \& (7.3))} \quad &\geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k .
\end{aligned}$$

This shows (7.117). $\square$

Lemma 7.44. *In case* **(t3–5)(cB)** *we get Configuration* $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$.

Proof. Recall that by Lemma 7.10 we know that $\mathcal{F}$, as defined in (7.13), is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover. We introduce another $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover,

$$\mathcal{F}' := \mathcal{F} \cup \{X \in \mathcal{V}(\mathcal{M}_B) \,:\, X \subseteq \mathfrak{A}\} \,.$$

By (7.27) and as we are in case (**cB**), we have $\deg^{\max}{}_{G_\nabla}(V_1(\mathcal{M}), \bigcup \mathcal{F}) \leqslant \frac{2\eta^3}{3\cdot 10^3}k$. Furthermore, as we are in case **(t3–5)**, we have $V_1(\mathcal{M}) \cap V_{\rightsquigarrow\mathfrak{A}} = \emptyset$. Thus,

$$\deg^{\max}{}_{G_\nabla}\left(V_1(\mathcal{M}), \bigcup \mathcal{F}'\right) \leqslant \frac{2\eta^3}{10^3}k. \tag{7.119}$$

We use Lemma 7.31 with the following input parameters: $r_{\triangleright\mathrm{L}7.31} := 2$, $\Omega_{\triangleright\mathrm{L}7.31} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.31} := \eta^4/10^{11}$, $\eta_{\triangleright\mathrm{L}7.31} := \rho/2\Omega^*$, $\delta_{\triangleright\mathrm{L}7.31} := \rho\eta^8/(10^{27}(\Omega^*)^3)$, $\varepsilon_{\triangleright\mathrm{L}7.31} := \bar{\varepsilon}$, $\mu_{\triangleright\mathrm{L}7.31} := \beta$ and $d_{\triangleright\mathrm{L}7.31} := \bar{d}$. We use the following vertex sets $Y_{\triangleright\mathrm{L}7.31} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, and $X_2 := V(\bar{\mathcal{M}}) \setminus \bigcup \mathcal{F}' \subseteq \bigcup \mathbf{V}^{\restriction 1}$. The partitions of X_0 and X_1 in Lemma 7.31 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_\mathcal{D})$.

Condition 1 of Lemma 7.31 follows from Lemma 7.42(b). Condition 2 follows by the assumption of Lemma 7.44 on the size of $V(\mathcal{M})$. Condition 4 follows from the definition of $\mathcal{M}$. Condition 5 holds since $V(\mathcal{M})$ does not meet $\mathbf{\Psi}$.

It remains to see Condition 3, for $i = 1$. For this, first note that from Lemma 7.10 we get that

$$\deg^{\min}{}_{G_\nabla}\left(V_1(\mathcal{M}), V^{\restriction 1}_{\mathrm{good}}\right) \overset{(\mathbf{cB})}{\geqslant} \deg^{\min}{}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V}), V^{\restriction 1}_{\mathrm{good}}\right) \geqslant \mathfrak{p}_1(1 + \frac{\eta}{20})k \,. \tag{7.120}$$

From this, we calculate that

$$\begin{aligned}
\deg^{\min}{}_{G_\mathcal{D}}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1}\big) &\geqslant \deg^{\min}{}_{G_\nabla}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1}\big) \\
&\quad - \deg^{\max}{}_{G_{\exp}}\big(V_1(\mathcal{M}), V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
\text{(by (7.9) \& (7.6))} \quad &\geqslant \deg^{\min}{}_{G_\nabla}\big(V_1(\mathcal{M}), V^{\restriction 1}_{\mathrm{good}}\big) \\
&\quad - \deg^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}), \mathfrak{A}\big) \\
&\quad - \deg^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}), \mathbb{L}_{\eta,k}(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
&\quad - \deg^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}), V(G_{\exp}) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
&\quad - \deg^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}), V(G_{\exp}) \cap V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \\
\text{(by (7.120), as } V_1(\mathcal{M}) \cap V_{\rightsquigarrow\mathfrak{A}} = \emptyset \text{ \& } (\mathbf{cB})) \quad &\geqslant \mathfrak{p}_1(1 + \frac{\eta}{20}k) - \frac{\rho k}{100\Omega^*} \\
&\quad - \deg^{\max}{}_{G_\nabla}\big(\mathbb{XA} \setminus \mathtt{P}_3, \mathbb{XA}\big) \\
&\quad - \deg^{\max}{}_{G_\nabla}(V_1(\mathcal{M}), V(G_{\exp})) \\
\text{(by def of } \mathtt{P}_3 \text{ \& as } V_1(\mathcal{M}) \cap \mathbf{shadow}_G(V(G_{\exp}), \rho k) = \emptyset \text{ by } \mathbf{(t3–5)}) \quad &\geqslant \mathfrak{p}_1(1 + \frac{\eta}{20})k - \frac{\rho k}{100\Omega^*} - \frac{\eta^3 k}{10^3} - \rho k.
\end{aligned} \tag{7.121}$$

We obtain

$$\begin{aligned}
\mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M})\setminus Y_{\triangleright \mathrm{L}7.31},X_2) &\geqslant \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}\big(V_1(\mathcal{M})\setminus Y_{\bar{M}},V(\bar{\mathcal{M}})\big)-\mathrm{deg}^{\max}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M}),\bigcup\mathcal{F}')\\
\text{(by def of } \bar{\mathcal{M}},\text{ (7.119))}\quad &\geqslant \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M})\setminus Y_{\bar{M}},V_{\mathrm{leftover}})-\frac{2\eta^3k}{10^3}\\
\text{(by (7.121) and by def of } Y_{\triangleright \mathrm{L}7.31})\quad &\geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k-\frac{\rho k}{100\Omega^*}-\frac{\eta^3k}{10^3}-\rho k-\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\
&\geqslant \mathfrak{p}_1(1+\frac{\eta}{30})k\ . \qquad (7.122)
\end{aligned}$$

Since the last term is greater than $\gamma_{\triangleright \mathrm{L}7.31}k=\frac{\eta^4}{10^{11}}k$ by (7.15), we see that Condition 3 of Lemma 7.31 is satisfied.

The output of Lemma 7.31 are three non-empty sets X_0',X_1',X_2' disjoint from $Y_{\triangleright \mathrm{L}7.31}$, together with $(4\bar{\varepsilon},\frac{\bar{d}}{4})$-super-regular pairs $\{Q_0^{(j)},Q_1^{(j)}\}_{j\in\mathcal{Y}}$ which cover (X_0',X_1') with the following properties.

$$\text{(by Lemma 7.31 (a))}\quad \min\left\{|Q_0^{(j)}|,|Q_1^{(j)}|\right\}\geqslant\frac{\beta k}{2}\text{ for each } j\in\mathcal{Y}\ , \qquad (7.123)$$

$$\text{(by Lemma 7.31 (b))}\quad \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}(X_2',X_1')\geqslant\delta_{\triangleright \mathrm{L}7.31}k\ , \qquad (7.124)$$

$$\begin{aligned}
\text{(by Lemma 7.31 (c) and (7.122))}\quad \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}(X_1',X_2') &\geqslant \mathfrak{p}_1(1+\frac{\eta}{30})k-\frac{\eta^4k}{2\cdot 10^{11}}\\
&\geqslant \mathfrak{p}_1(1+\frac{\eta}{40})k\ .
\end{aligned}\qquad (7.125)$$

We now verify that the sets X_0',X_1',X_2', the semiregular matching $\mathcal{N}_{\triangleright \mathrm{D}7.24}:=\bar{\mathcal{M}}$ together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$, and the family $\{(Q_0^{(j)},Q_0^{(j)})\}_{j\in\mathcal{Y}}$ satisfy all the conditions of Configuration $(\diamond\mathbf{9})(\delta_{\triangleright \mathrm{L}7.31},\frac{2\eta^3}{10^3},\mathfrak{p}_1(1+\frac{\eta}{40})k,\mathfrak{p}_2(1+\frac{\eta}{20})k,\frac{400\varepsilon}{\eta},\frac{d}{2},\frac{\eta\pi\mathfrak{c}}{200k},4\pi,\gamma^3\rho/32\Omega^*,\eta^2\nu/2\cdot 10^4)$.

By Lemma 7.10, since we are in case (**cB**) and by (7.119), the pair X_0',X_1' together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$ witnesses Preconfiguration $(\heartsuit\mathbf{1})(\frac{2\eta^3}{10^3},\mathfrak{p}_2(1+\frac{\eta}{20})k)$. By Lemma 7.42 (a), $\bar{\mathcal{M}}$ is as required for Configuration $(\diamond\mathbf{9})$.

To see that G is in Preconfiguration (**reg**)$(4\pi,\gamma^3\rho/32\Omega^*,\eta^2\nu/2\cdot 10^4)$, note that $4\bar{\varepsilon}\leqslant 4\pi$ and $\bar{d}/4\geqslant\gamma^3\rho/32\Omega^*$ (in both cases **(M1)** and **(M1)**). Further, Property (7.47) follows from (7.123) since $\beta/2\geqslant\eta^2\nu/2\cdot 10^4$.

Finally, by definition of X_2, the set X_2' is as required, with Property (7.63) following from (7.125), and Property (7.64) following from (7.124). □

We are now reaching the last lemma of this section, dealing with the last remaining case.

Lemma 7.45. *In Case* (**t5**)(**cA**) *we get Configuration* $(\diamond\mathbf{10})\big(\varepsilon,\frac{\gamma^2 d}{2},\pi\sqrt{\varepsilon'}\nu k,\frac{(\Omega^*)^2k}{\gamma^2},\frac{\eta}{40}\big)$.

Proof. Since we are in case (**t5**), we have $V(\mathcal{M})\subseteq V(G_{\mathrm{reg}})$. Therefore,

$$\begin{aligned}
\mathrm{deg}^{\min}{}_{G_{\mathrm{reg}}}(V(\mathcal{M}),V_{\mathrm{good}}) &\geqslant \mathrm{deg}^{\min}{}_{G_{\nabla}}(V(\mathcal{M}),V_+\setminus L_\#)-\mathrm{deg}^{\max}{}_{G_{\nabla}}(V(\mathcal{M}),\mathbf{\Psi})\\
&\quad -\mathrm{deg}^{\max}{}_{G_{\nabla}}(V(\mathcal{M}),\mathfrak{A})-\mathrm{deg}^{\max}{}_{G_{\nabla}}(V(\mathcal{M}),V(G_{\exp}))\\
&\geqslant (1+\frac{\eta}{20})k, \qquad (7.126)
\end{aligned}$$

where the last line follows as $V(\mathcal{M}) \subseteq \mathbb{XA} \setminus \mathtt{P} \subseteq \mathbb{YA} \setminus V_{\not\leadsto\boldsymbol{\Psi}}$ by (**cA**) and furthermore, $V(\mathcal{M}) \cap (\mathbf{shadow}_G(V(G_{\mathrm{exp}}), \rho k) \cup V_{\leadsto\mathfrak{A}}) = \emptyset$ by (**t5**).

Define

$$\begin{aligned}\mathcal{C} &:= \left\{C \setminus \left(L_{\#} \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V_{\not\leadsto\boldsymbol{\Psi}} \cup \mathtt{P}_1\right) \,:\, C \in \mathbf{V}\right\} ,\\ \mathcal{C}^- &:= \left\{C \in \mathcal{C} \,:\, |C| < \sqrt{\varepsilon'}\mathfrak{c}\right\} ,\end{aligned}$$

We have

$$\left|\bigcup \mathcal{C}^-\right| \leqslant \sum_{C \in \mathcal{C}} \sqrt{\varepsilon'}|C| \leqslant \sqrt{\varepsilon'}n \,. \tag{7.127}$$

Set $\mathcal{V}^\circ := \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B) \cup (\mathcal{C} \setminus \mathcal{C}^-)$ and let G° be the subgraph of G with vertex set $\bigcup \mathcal{V}^\circ$ and all edges from $E(G_{\mathrm{reg}})$ induced by $\bigcup \mathcal{V}^\circ$ plus all edges of $E(G_\nabla) \setminus E(G_{\mathrm{exp}})$ between X and Y for all $(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B$. Apply Fact 2.7 (and recall Definition 4.7 (2)) to see that each pair of sets $X, Y \in \mathcal{V}^\circ$ forms an ε-regular pair of density either 0 or at least $\gamma^2 d/2$ (whose edges either lie in G_{reg} or touch $\mathfrak{A}$).

Next, observe that from Setting 7.4 (3), Fact 4.3 and Fact 4.4, and using Definition 4.7(6), we find that for all $X \in \mathcal{V}^\circ$ which lie in some cluster of $\mathbf{V}$, we have $|\bigcup \mathrm{N}_{G^\circ}(X)| \leqslant |\bigcup \mathrm{N}_{G_{\mathcal{D}}}(X)| \leqslant \frac{\Omega^*}{\gamma} \cdot \frac{\Omega^* k}{\gamma}$. Also, observe that for all $X \in \mathcal{V}^\circ$ which do not lie in some cluster of $\mathbf{V}$, we know from Setting 7.4 (4) that X does not see any edges from $E(G_{\mathrm{reg}})$. This means that $\bigcup \mathrm{N}_{G^\circ}(X)$ is contained in the partner of X in $\mathcal{M}_A \cup M_B$ (which has size at most $\mathfrak{c} \leqslant \varepsilon' k$ by Setting 7.4 (4) and Definition 4.7 (3)).

Thus we obtain that

$$(G^\circ, \mathcal{V}^\circ) \text{ is an } (\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\mathfrak{c}, \frac{(\Omega^*)^2 k}{\gamma^2})\text{-regularized graph.} \tag{7.128}$$

Define

$$\mathcal{L}^\circ := \left\{X \in \mathcal{V}^\circ \setminus \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B) \,:\, \mathrm{deg^{min}}_{G^\circ}(X) \geqslant (1 + \frac{\eta}{2})k\right\}.$$

We claim that the following holds.

Claim 7.45.1. There are distinct $X_A, X_B \in \mathcal{V}^\circ$, with $E(G^\circ[X_A, X_B]) \neq \emptyset$, such that we have $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k$ for all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in X_A$, and all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in X_B$.

Then, setting $\tilde{G}_{\triangleright\mathrm{D}7.26} := G^\circ$, $\mathcal{V}_{\triangleright\mathrm{D}7.26} := \mathcal{V}^\circ$, $\mathcal{M}_{\triangleright\mathrm{D}7.26} := \mathcal{M}_A \cup \mathcal{M}_B$, $\mathcal{L}^*_{\triangleright\mathrm{D}7.26} := \mathcal{L}^\circ$, $A_{\triangleright\mathrm{D}7.26} := X_A$, and $B_{\triangleright\mathrm{D}7.26} := X_B$, we have obtained Configuration $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \eta/40\big)$. Indeed, using (7.128), and the definition of $\mathcal{L}^\circ$ we see that $(\tilde{G}_{\triangleright\mathrm{D}7.26}, \mathcal{V}_{\triangleright\mathrm{D}7.26})$, $\mathcal{M}_{\triangleright\mathrm{D}7.26}$ and $\mathcal{L}^*_{\triangleright\mathrm{D}7.26}$ are as desired and fulfil (c). Claim 7.45.1 together with the fact that $\deg_{G^\circ}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ)$ for all $v \in V(G^\circ)$ ensure that also (a) and (b) hold.

It only remains to prove Claim 7.45.1.

Proof of Claim 7.45.1. In order to find X_A and X_B as in the statement of the lemma, we shall exploit the matching $\mathcal{M}$; the relation between $\mathcal{M}$ and $(G^\circ, \mathcal{V}^\circ)$, $\mathcal{M}_A \cup \mathcal{M}_B$, and $\mathcal{L}^\circ$ is not direct. We proceed as follows. In Subclaim 7.45.1.1 we find a suitable $\mathcal{M}$-edge. In case (**M1**) this $\mathcal{M}$-edge gives readily a suitable pair $(A_{\triangleright \mathrm{D}7.26}, B_{\triangleright \mathrm{D}7.26})$. In case (**M2**) we have to work on the $\mathcal{M}$-edge to get a suitable $\mathbf{G}_{\mathrm{reg}}$-edge, this will be done in Subclaim 7.45.1.2. Only then do we find $(A_{\triangleright \mathrm{D}7.26}, B_{\triangleright \mathrm{D}7.26})$.

Subclaim 7.45.1.1. There is an $\mathcal{M}$-edge (A, B) such that $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{200}$ for at least $|A|/2$ vertices $v \in A$, and at least $|B|/2$ vertices $v \in B$.

Proof of Subclaim 7.45.1.1. Set $S := \mathbf{shadow}_{G_{\mathrm{reg}}}(\bigcup \mathcal{C}^-, \frac{\eta k}{200})$, and note that by Fact 7.1 we have $|S| \leqslant |\bigcup \mathcal{C}^-| \cdot \frac{200\Omega^*}{\eta}$. So, setting $\mathcal{M}_S := \{(X, Y) \in \mathcal{M} \,:\, |(X \cup Y) \cap S| \geqslant |X \cup Y|/4\}$ we find that

$$|V(\mathcal{M}_S)| \;\leqslant\; 4|S| \;\overset{(7.127)}{\leqslant}\; \frac{800\sqrt{\varepsilon'}\Omega^* n}{\eta} \;<\; \frac{\rho n}{\Omega^*} \;\leqslant\; |V(\mathcal{M})|,$$

where the last inequality holds by assumption of Lemma 7.45. Consequently, $\mathcal{M} \neq \mathcal{M}_S$.

Let $(A, B) \in \mathcal{M} \setminus \mathcal{M}_S$. We will show that (A, B) satisfies the requirements of the subclaim. We start by proving that

$$V_+ \cap V(G^\circ) \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \subseteq V(G_{\mathrm{exp}}) \cup (V_{\rightsquigarrow \mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)). \tag{7.129}$$

Indeed, observe that by (7.7),

$$\begin{aligned} V_+ \cap V(G^\circ) &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\mathrm{exp}}) \cup \left(\mathbb{L}_{\eta,k}(G) \setminus (L_\# \cup V_{\not\rightsquigarrow \mathbf{\Psi}} \cup \mathtt{P}_1)\right) \\ &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\mathrm{exp}}) \cup \left(\mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V_{\not\rightsquigarrow \mathbf{\Psi}} \cup \mathtt{P}_1)\right) . \end{aligned}$$

So, in order to show (7.129), it suffices to see that for each $X \in \mathcal{V}^\circ \setminus \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B)$ with $X \subseteq \mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V_{\not\rightsquigarrow \mathbf{\Psi}} \cup \mathtt{P}_1 \cup V(G_{\mathrm{exp}}) \cup V_{\rightsquigarrow \mathfrak{A}})$ we have $X \in \mathcal{L}^\circ$. So assume X is as above. Let $v \in X$. We calculate

$$\begin{aligned} \deg_{G_{\mathrm{reg}}}(v, V(G^\circ)) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B)) \\ \scriptstyle{(v \notin V(G_{\mathrm{exp}}))} \quad &\geqslant (1 + \frac{9\eta}{10})k - \deg_G(v, \mathbf{\Psi}) - \deg_{G_{\mathcal{D}}}(v, \mathfrak{A}) \\ &\quad - \deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \\ \scriptstyle{(v \notin V_{\not\rightsquigarrow \mathbf{\Psi}} \cup V_{\rightsquigarrow \mathfrak{A}} \cup \mathtt{P}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B))} \quad &\geqslant (1 + \frac{9\eta}{10})k - \frac{\eta k}{100} - \frac{\rho k}{100\Omega^*} - \gamma k \\ &\geqslant (1 + \frac{\eta}{2})k . \end{aligned}$$

We deduce that $X \in \mathcal{L}^\circ$, which finishes the proof of (7.129).

Next, observe that by the definition of $\mathcal{C}$, we have

$$\begin{aligned} V_+ \cap V(G^\circ) &\supseteq V_{\mathrm{good}} \cap V(G^\circ) \\ &\supseteq V_{\mathrm{good}} \setminus \left(V_{\mathrm{good}} \setminus V(G^\circ)\right) \\ &\supseteq V_{\mathrm{good}} \setminus (V_{\not\rightsquigarrow \mathbf{\Psi}} \cup \mathtt{P}_1 \cup \bigcup \mathcal{C}^- \cup \mathfrak{A} \cup V(G_{\mathrm{exp}})). \end{aligned} \tag{7.130}$$

We are now ready to prove Subclaim 7.45.1.1. For each vertex $v \in A \setminus S$, we have

$$\begin{aligned}
\deg_{G_{\mathrm{reg}}}\left(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ\right) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V_+ \cap V(G^\circ)) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, (V_+ \cap V(G^\circ)) \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \mathcal{L}^\circ)) \\
\text{(by (7.130), (7.129))} \qquad &\geqslant \deg_{G_{\mathrm{reg}}}(v, V_{\mathrm{good}}) - \deg_{G_{\mathrm{reg}}}(v, V_{\not\leadsto\Psi} \cup \mathtt{P}_1 \cup \bigcup \mathcal{C}^-) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, \mathfrak{A}) - 2\deg_{G_{\mathrm{reg}}}(v, V(G_{\mathrm{exp}})) \\
&\quad - \deg_{G_{\mathrm{reg}}}\left(v, (V_{\leadsto\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\right) \\
\text{(by (7.126), as } v \notin S \cup \mathtt{P}\text{, by } (\mathbf{t5})) \qquad &\geqslant (1 + \frac{\eta}{20})k - \frac{\eta^2 k}{10^5} - \frac{\eta k}{200} - \frac{\rho k}{100\Omega^*} - 2\rho k - \frac{2\eta^2 k}{10^5} \\
&> (1 + \frac{\eta}{40})k + \frac{\eta k}{200}\ ,
\end{aligned}$$

where for the second to last inequality we used the abreviation 'by $(\mathbf{t5})$' to indicate that this case implies that $v \notin \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup \mathbf{shadow}_{G_\nabla}((V_{\leadsto\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup M_B), \frac{2\eta^2 k}{10^5})$. As $|A \setminus S| \geqslant |A|/2$, we note that the set A fulfils the requirements of the claim.

The same calculations hold for B. This finishes the proof of Subclaim 7.45.1.1. □

The next auxiliary subclaim is needed in our proof of Claim 7.45.1 in case **(M2)**.

Subclaim 7.45.1.2. Suppose that case **(M2)** occurs. Then there exists an edge $C_A C_B \in E(\mathbf{G}_{\mathrm{reg}})$ such that $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{400}$ for all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in C_A$, and all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in C_B$. Moreover, there exist $A, B \in \mathcal{V}(\mathcal{M})$ such that $|C_A \cap A| > \sqrt{\varepsilon'}\mathfrak{c}$ and $|C_B \cap B| > \sqrt{\varepsilon'}\mathfrak{c}$.

Proof of Claim 7.45.1.2. Let $(A, B) \in \mathcal{M}$ be given as in Subclaim 7.45.1.1. Let $P_A \subseteq A$, and $P_B \subseteq B$ be the vertices which fail the assertion of Subclaim 7.45.1.1. Note that with this notation, Subclaim 7.45.1.1 states that

$$|A \setminus P_A| \geqslant |A|/2. \tag{7.131}$$

Call a cluster $C \in \mathbf{V}$ *A-negligible* if $|C \cap (A \setminus P_A)| \leqslant \frac{\gamma^3\mathfrak{c}}{16\Omega^* k}|A|$. Let R_A be the union of all A-negligible clusters.

Recall that (A, B) is entirely contained in one dense spot from $(U, W; F) \in \mathcal{D}_\nabla$ (cf. **(M2)**). So by Fact 4.3, and since the spots in $\mathcal{D}_\nabla$ are $(\frac{\gamma^3 k}{4}, \frac{\gamma^3 k}{4})$-dense, we know that $\max\{|U|, |W|\} \leqslant \frac{4\Omega^* k}{\gamma^3}$. In particular, there are at most $\frac{4\Omega^* k}{\gamma^3\mathfrak{c}}$ A-negligible clusters which intersect to $A \cap R_A$.

As these clusters are all disjoint, we find that

$$|(A \cap R_A) \setminus P_A| \leqslant \frac{4\Omega^* k}{\gamma^3\mathfrak{c}} \cdot |C \cap (A \setminus P_A)| \leqslant \frac{|A|}{4}.$$

This gives

$$|A \setminus (P_A \cup R_A)| \overset{(7.131)}{\geqslant} \frac{|A|}{2} - |(A \cap R_A) \setminus P_A| \geqslant \frac{|A|}{4}\ .$$

Similarly, we can introduce the notion B-negligible clusters, and the set R_B, and get $|(B \cap R_B) \setminus P_B| \leqslant \frac{|B|}{4}$ and $|B \setminus (P_B \cup R_B)| \geqslant |B|/4$.

By the regularity of the pair (A,B) there exists at least one edge $ab \in E\big(G^*[A \setminus (P_A \cup R_A), B \setminus (P_B \cup R_B)]\big)$, where $a \in A, b \in B$, and G^* is the graph formed by edges of $\mathcal{D}_\nabla$. As $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}})$ by the assumption of case **(t5)**, we have that $ab \in E(G_{\mathrm{reg}})$. Let $C_A, C_B \in \mathbf{V}$ be the clusters containing a and b, respectively. Note that $C_A C_B \in E(\mathbf{G}_{\mathrm{reg}})$.

Now as $a \notin R_A$, also C_A is disjoint from R_A, and thus

$$|C_A \cap (A \setminus P_A)| > \frac{\gamma^3 \mathfrak{c}}{16\Omega^* k} \cdot \frac{\widehat{\alpha}\rho k}{\Omega^*} > \sqrt{\varepsilon'}\mathfrak{c} \,.$$

This proves the "moreover" part of the claim for C_A. So there are at least $2\varepsilon'\mathfrak{c}$ vertices v in C_A with $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1+\frac{\eta}{40})k + \frac{\eta k}{200}$ (by the definition of P_A). By Lemma 2.10, and using Facts 4.3 and 4.4, we thus have that $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1+\frac{\eta}{40})k + \frac{\eta k}{400}$ for all but at most $2\varepsilon'\mathfrak{c}$ vertices v of C_A. The same calculations hold for C_B. $\square$

In the remainder of the proof of Claim 7.45.1 we have to distiguish between cases **(M1)** and **(M2)**.

Let us first consider the case **(M2)**. Let $C_A, C_B \in \mathbf{V}$ and $A, B \in \mathcal{V}(\mathcal{M})$ be given by Subclaim 7.45.1.2. We have $|C_A \setminus (V_{\not\leadsto\Psi} \cup L_\# \cup \mathtt{P}_1)| > \sqrt{\varepsilon'}|C_A|$ by Subclaim 7.45.1.2 and by the definition of $\mathcal{M}$ and the definition of $\mathtt{P}$. Thus, $C_A \cap V(G^\circ)$ is non-empty. Let $X_A \in \mathcal{V}^\circ$ be an arbitrary set in C_A. Similarly, we obtain a set $X_B \in \mathcal{V}^\circ$, $X_B \subseteq C_B$. The claimed properties of the pair (X_A, X_B) follow directly from Subclaim 7.45.1.2.

It remains to treat the case **(M1)**. Let (A,B) be from Subclaim 7.45.1.1. Let $(X_A, X_B) \in \mathcal{M}_{\mathrm{good}}$ be such that $X_A \supseteq A$ and $X_B \supseteq B$. Claim 7.45.1.1 asserts that at least

$$\frac{|A|}{2} \overset{\textbf{(M1)}}{\geqslant} \frac{\eta^2\mathfrak{c}}{2\cdot 10^4} > 2\varepsilon'\mathfrak{c}$$

vertices of A have large degree (in G_{reg}) into the set $V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ$. Therefore, by Lemma 2.10, X_A and X_B satisfy the assertion of the Claim.

This proves Claim 7.45.1, and thus finishes the proof of Lemma 7.45. $\square$

$\square$

The proof of Lemma 7.35 follows by putting together Lemmas 7.40, 7.41, 7.43, 7.44, and 7.45.

8 Embedding trees

In this section we provide an embedding of a tree $T_{\triangleright\mathrm{T}1.3} \in \mathbf{trees}(k)$ in the setting of the configurations introduced in Section 7. In Section 8.1 we first give a fairly detailed overview of the embedding techniques used. In Section 8.2 we introduce a class of stochastic processes which will be used for some embeddings. Section 8.3 contains a number of lemmas about embedding small trees, and use them for embedding knags and shrubs of a given fine partition of $T_{\triangleright\mathrm{T}1.3}$. Embedding the entire tree $T_{\triangleright\mathrm{T}1.3}$ is then handled in the final Section 8.4. There we have to distinguish between particular configurations. The configurations are grouped into three categories (Section 8.4.1, Section 8.4.2, and Section 8.4.3) corresponding to the similarities between the configurations.

8.1 Overview of the embedding procedures

Given a host graph $G_{\triangleright\mathrm{T1.3}}$ with one of the Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{10})$, we have to embed in it a given tree $T = T_{\triangleright\mathrm{T1.3}} \in \mathbf{trees}(k)$, which comes with its τk-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. The τk-fine partition of T will make it possible to combine embeddings of smaller parts of T into one embedding of the whole tree. This means that we will first develop tools for embedding singular shrubs and knags of the τk-fine partition into various basic building bricks of the configurations: the avoiding set $\mathfrak{A}$, the expander G_{exp}, regular pairs, and vertices of huge degree $\mathbf{\Psi}$. Second, we will combine these basic techniques to embed the entire tree T. Here, the order in which different parts of T are embedded is important. Also, it will be crucial at some points to reserve places for parts of the tree which will be embedded only later.

In the following subsections, we draft our embedding techniques. We group them into five categories comprising of related configurations[23]: Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$, Configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{7})$, Configuration $(\diamond\mathbf{8})$, Configuration $(\diamond\mathbf{9})$, and Configuration $(\diamond\mathbf{10})$, treated in Sections 8.1.1, 8.1.2, 8.1.3, 8.1.4, 8.1.5, respectively.

8.1.1 Embedding overview for Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$

Recall that we are working under Setting 7.4. In each of the Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$ we have sets $\mathbf{\Psi}', \mathbf{\Psi}'', L'', L'$ and V_1. Further, we have some additional sets (V_2 and/or $\mathfrak{A}'$) depending on the particular configuration.

A common embedding scheme for Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$ is illustrated in Figure 8.1. There are two stages of the embedding procedure: the knags, the shrubs $\mathcal{S}_A$ and some parts of the shrubs $\mathcal{S}_B$ are embedded in Stage 1, and then in Stage 2 the remainders of $\mathcal{S}_B$ are embedded. Recall that $\mathcal{S}_A$ contains both internal and end shrubs while $\mathcal{S}_B$ contains exclusively end shrubs. We note that here the shrubs $\mathcal{S}_B$ are further subdivided and some parts of them are embedded in the Stage 1 and some in Stage 2.

- In Stage 1, the knags of T are embedded in $\mathbf{\Psi}''$ and V_1 so that W_A is mapped to $\mathbf{\Psi}''$ and W_B is mapped to V_1.
- In Stage 1, the internal and end shrubs of $\mathcal{S}_A$ are embedded using the sets V_1, V_2 and $\mathfrak{A}'$ which are specific to the particular Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$. The vertices of $\mathcal{S}_A$ neighbouring W_A are always embedded in V_1. Parts of the shrubs $\mathcal{S}_B$ are embedded while the ancestors of the unembedded remainders are embedded on vertices which have large degrees in $\mathbf{\Psi}'$.
- In Stage 2, the embedding of $\mathcal{S}_B$ is finalized. The remainders of $\mathcal{S}_B$ are embedded starting with embedding their roots in $\mathbf{\Psi}'$.

A hierarchy of the embedding lemmas used to resolve Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$ is given in Table 8.1.

[23] Configuration $(\diamond\mathbf{1})$ is trivial (see Section 8.4.1) and needs no draft.

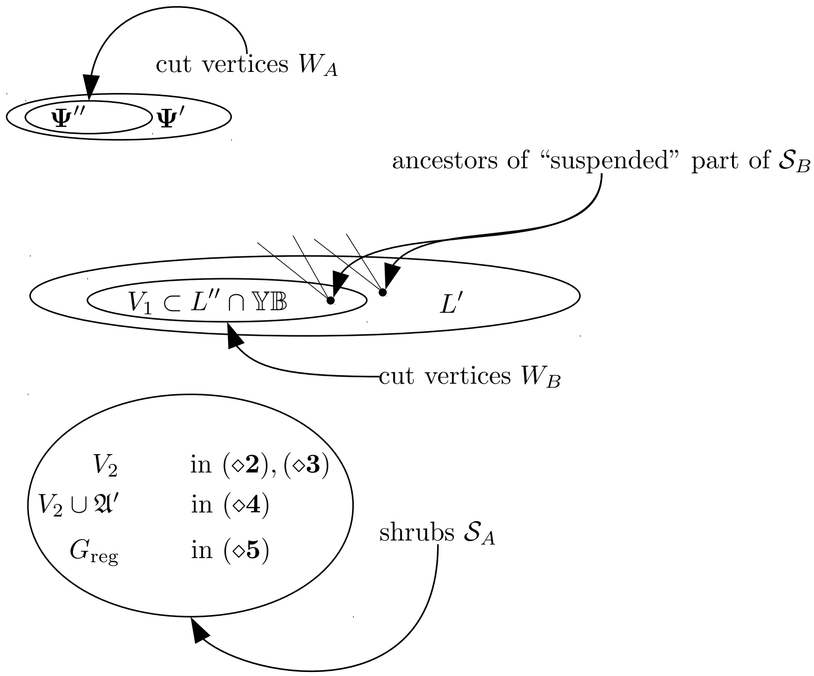

Figure 8.1: An overview of embedding of a tree $T \in \mathbf{trees}(k)$ given with its fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ using Configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$). The knags are embedded between $\mathbf{\Psi}''$ and V_1, all the shrubs $\mathcal{S}_A$ are embedded into sets specific to particular configurations so that the vertices neighbouring W_A are embedded in V_1. Parts of the shrubs $\mathcal{S}_B$ are embedded directly (using various embedding techniques), while the rest is "suspended", i.e., the ancestors of the unembedded remainders are embedded on vertices which have large degrees in $\mathbf{\Psi}'$. The embedding of $\mathcal{S}_B$ is then finalized in the last stage.

8.1.2 Embedding overview for Configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{7}$)

Suppose Setting 7.4 and 7.7 (see Remark 8.1 below for a comment on the constants $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$). Recall that we have in each of these configurations sets $V_0 \cup V_1 \subseteq \mathfrak{P}_0$, sets $V_2 \cup V_3 \subseteq \mathfrak{P}_1$ and $V_{\mathrm{good}}^{\restriction 2}$.

A common embedding scheme for Configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{7}$) is illustrated in Figure 8.2. The embedding has three parts.

- The knags of T are embedded between V_0 and V_1 so that W_A is mapped to V_1 and W_B is mapped to V_0 using either the Preconfiguration (**exp**) or (**reg**). Thus $W_A \cup W_B$ ar mapped to $\subseteq \mathfrak{P}_0$.

- The internal shrubs of T are embedded in $V_2 \cup V_3$, always putting neighbours of W_A into V_2. Note that the internal shrubs are therefore embedded in $\mathfrak{P}_1$, and thus there is no interference with embedding the knags. We need to understand why a mere degree of δk (from V_1 to V_2, ensured by (7.48) and (7.52), with $\delta \ll 1$) is sufficient for embedding internal shrubs of potentially big total order, that is, how to ensure that already embedded internal trees do

Main embedding lemma: Lemma 8.18		
$\Uparrow$	$\Uparrow$	$\Uparrow$
Shrubs $\mathcal{S}_A$ ($\diamond$**2**): Lemma 8.4 ($\diamond$**3**): Lemma 8.13 ($\diamond$**4**): Lemma 8.14 ($\diamond$**5**): regularity	Shrubs $\mathcal{S}_B$ (Stage 1): Lemma 8.17	Shrubs $\mathcal{S}_B$ (Stage 2): Lemma 8.16

Table 8.1: Embedding lemmas employed for Configurations ($\diamond$**2**)–($\diamond$**5**).

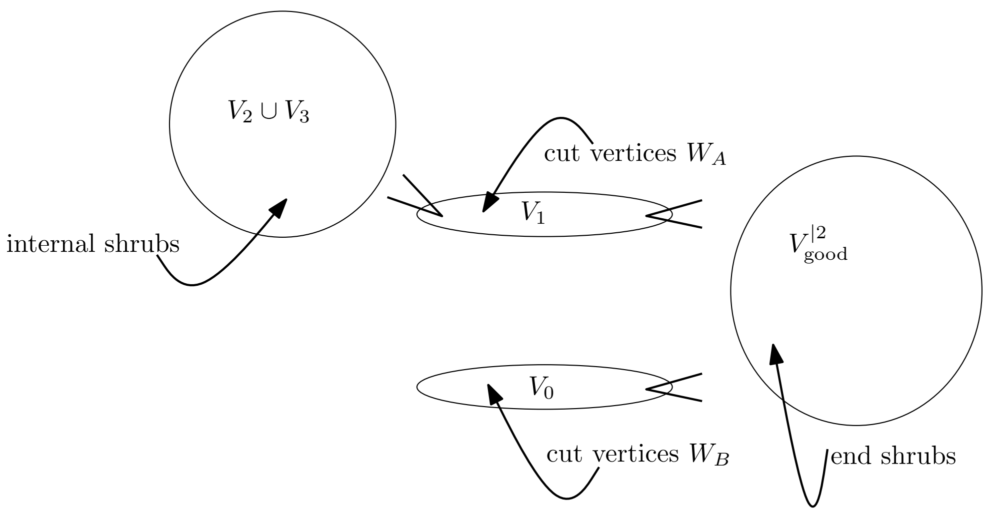

Figure 8.2: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configurations ($\diamond$**6**)–($\diamond$**7**). The knags are embedded between V_0 and V_1, the internal shrubs are embedded in $V_2 \cup V_3$, and the end shrubs are embedded using $V_{\mathrm{good}}^{\restriction 2}$.

not cause a blockage later. Here the expansion[24] ruling between the V_2 and V_3 comes into play. This property (together with other properties of Preconfigurations (**exp**) and (**reg**)) will allow that, once finished embedding an internal tree, the follow-up knag can be embedded in a place (in V_1) which sees very little of the previously embedded internal shrubs.

This is the only part of the embedding process which makes use of the specifics of Configurations ($\diamond$**6**) and ($\diamond$**7**). For this reason we will be able to follow the same embedding scheme as presented here also for Configuration ($\diamond$**8**), the only difference being the embedding of the internal shrubs (see Section 8.1.3).

- The end shrubs are embedded in the yet unoccupied part of G. For this we use the properties of Preconfigurations ($\heartsuit$**1**) or ($\heartsuit$**2**). The end shrubs are embedded using (but not entirely into) the designated vertex set $V_{\mathrm{good}}^{\restriction 2}$.

[24]This expansion is given by the presence of G_{exp} in Configurations ($\diamond$**6**) (cf. (7.50)–(7.51)), and by the presence of the avoiding set $\mathfrak{A}$ in Configurations ($\diamond$**7**) ($V_2 \subseteq \mathfrak{A}^{\restriction 1} \setminus \bar{V}$).

Main embedding lemma: Lemma 8.23

$\Uparrow$ $\Uparrow$

Internal part	End shrubs
($\diamond$**6**), ($\diamond$**7**): Lemma 8.19	($\heartsuit$**1**): Lemma 8.21
($\diamond$**8**): Lemma 8.20	($\heartsuit$**2**): Lemma 8.22

$\Uparrow$ $\Uparrow$

Knags	Internal shrubs
(**exp**): Lemma 8.4	($\diamond$**6**): Lemma 8.11
(**reg**): Lemma 8.7	($\diamond$**7**): Lemma 8.12
	($\diamond$**8**): Lemmas 8.12, 8.8, 8.5

Table 8.2: Embedding lemmas employed for Configurations ($\diamond$**6**)–($\diamond$**8**) when embedding a tree $T \in \mathbf{trees}(k)$ with a given fine partition.

The above embedding scheme is divided in two main steps: first the knags and the internal trees are embedded (see Lemma 8.19), and this partial embedding is then extended to end shrubs (see Lemmas 8.21 and 8.22). A more detailed hierarchy of the embedding lemmas which are used is given in Table 8.2.

Remark 8.1. *In our application of Lemma 7.34 the number $\mathfrak{p}_1$ will be approximately the proportion of the total order of the internal shrubs of a given fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T while $\mathfrak{p}_2$ will be approximately the proportion of the total order of the end shrubs. The number $\mathfrak{p}_0$ is just a small constant.*

These numbers – scaled up by k – determine the parameter $h_1 \approx \mathfrak{p}_1 k$ (in Configurations ($\diamond$**8**) *and* ($\diamond$**9**)*) and $h_2 \approx \mathfrak{p}_2 k$ (in Configurations* ($\diamond$**6**)*–*($\diamond$**9**)*). The properties of these configurations will then allow to embed all the internal shrubs and end shrubs. Note that the parameter h_1 does not appear in Configurations* ($\diamond$**6**) *and* ($\diamond$**7**)*. This suggests that the total order of the internal shrubs is not at all important in Configurations* ($\diamond$**6**)*–*($\diamond$**7**)*. Indeed, we would succeed even embedding a tree with internal shrubs of total order say $100k$.*[25]

In view of this it might be tempting to think that the end shrubs in $\mathcal{S}_A$ could also be embedded using the same technique as the internal shrubs into the sets $V_2 \cup V_3$ provided by these configurations (cf. Figure 8.2). This is however not the case. Indeed, the minimum degree conditions (7.48), (7.52), *and* (7.56) *allow embedding only a small number of shrubs from a single cut-vertex $x \in W_A$ while there may be many end shrubs attached to x; cf. Remark 3.5*(ii)*.*

8.1.3 Embedding overview for Configuration ($\diamond$8)

Suppose Setting 7.4 and 7.7. We are working with sets V_0, V_1, $V_{\mathrm{good}}^{\restriction 2}$, V_2, V_3 and V_4 and with semiregular matching $\mathcal{N}$ coming from the configuration.

[25] Configuration ($\diamond$**8**) has this property only in part. We would succeed even embedding a tree with principal subshrubs of total order say $100k$ provided that the total order of peripheral subshrubs is somewhat smaller than h_1.

The embedding scheme follows Table 8.2, and is illustrated in Figure 8.3. Embedding of the

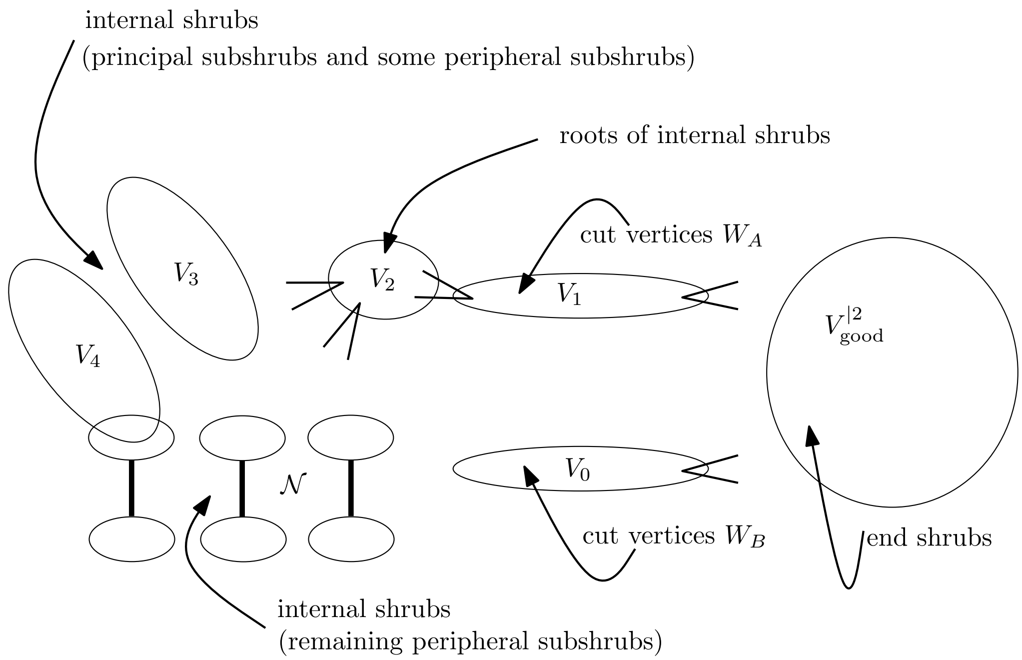

Figure 8.3: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configuration ($\diamond$**8**). The knags are embedded between V_0 and V_1. The roots of the internal shrubs are embedded in V_2. Some of the subshrubs of the internal shrubs are embedded in $V_3 \cup V_4$ and some in $\mathcal{N}$; principal subshrubs are always embedded in $V_3 \cup V_4$. The end shrubs are embedded in using $V_{\text{good}}^{\restriction 2}$.

knags and of the external shrubs is done in the same way as in Configurations ($\diamond$**6**)–($\diamond$**7**). We only describe here the way the internal shrubs are embedded. Their roots are embedded in V_2. From that point we proceed embedding subshrub by subshrub. Some of the subshrubs get embedded between V_3 and V_4. This pair of sets has the same expansion property as the pair V_2, V_3 in Configuration ($\diamond$**7**). In particular, it allows to avoid the shadow of the already occupied set so that the follow-up knag can be embedded in location almost isolated from the previous images, similarly as described in Section 8.1.2. For this reason we make sure that principal subshrubs get embedded here. The degree condition from V_2 to V_3 is too weak to ensure that all remaining subshrubs are embedded between V_3 and V_4. Therefore we might have to embed some subshrubs in $\mathcal{N}$. Condition (7.62) — where h_1 is approximately the order of the internal shrubs, as in Remark 8.1 — indicates that it should be possible to accommodate all the subshrubs. For technical reasons, the order in which different types of subshrubs are embedded is very important.

8.1.4 Embedding overview for Configuration ($\diamond$9)

The embedding process in Configuration ($\diamond$**9**) follows the same scheme as in Configurations ($\diamond$**6**)–($\diamond$**8**), but the embedding of the internal shrubs follows the regularity method. Assuming the simplest

situation $\mathcal{F} = \mathcal{V}_2(\mathcal{N})$ and $V_2 = V_1(\mathcal{N})$, we would have $\deg^{\min}{}_{G_{\mathrm{reg}}}(V_1, V_1(\mathcal{N})) \geqslant h_1$ (cf. (7.63)). See Figure 8.4 for an illustration. Similarly as above, the knags are embedded between V_0 and V_1.

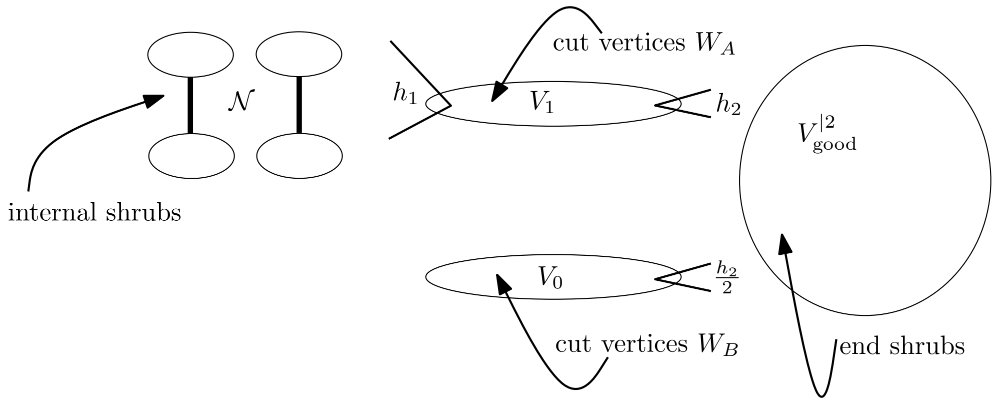

Figure 8.4: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configuration ($\diamond$**9**). The knags are embedded between V_0 and V_1, the internal shrubs using the regularity method in $\mathcal{N}$ and the end shrubs are embedded using $V^{\restriction 2}_{\mathrm{good}}$.

The internal shrubs are accommodated using the regularity method in $\mathcal{N}$, and the end shrubs are embedded in $V^{\restriction 2}_{\mathrm{good}}$ using Preconfiguration ($\heartsuit$**1**). The embedding lemma for this configuration is given in Lemma 8.24.

8.1.5 Embedding overview for Configuration ($\diamond$10)

Configuration ($\diamond$**10**) is very closely related to the structure obtained by Piguet and Stein [PS12] in their solution of the dense approximate case of Conjecture 1.2. Let us describe their proof first. Piguet and Stein prove that when $k > qn$ (for some fixed $q > 0$ and k sufficiently large) the cluster graph[26] $\mathbf{G}_{\mathrm{reg}}$ of a graph $G \in \mathbf{LKS}(n, k, \eta)$ contains the following structure (cf. [PS12, Lemma 7]). There is a set of clusters $\mathbf{L} \subseteq \mathbf{V}$ such that each cluster in $\mathbf{L}$ contains only vertices of captured degrees at least $(1 + \frac{\eta}{2})k$. There is a matching $M \subseteq \mathbf{G}_{\mathrm{reg}}$, and an edge AB, with $A, B \in \mathbf{L}$. One of the following conditions is satisfied

(H1) M covers $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(\{A, B\})$, or

(H2) M covers $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$, and the vertices in B have captured degrees at least $(1 + \frac{\eta}{2})\frac{k}{2}$ into $\bigcup(\mathbf{L} \cup V(M))$. Further, each edge in M has at most one endvertex in $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$.

Piguet and Stein use structures **(H1)** and **(H2)** to embed any given tree $T \in \mathbf{trees}(k)$ into G using the regularity method; see Sections 3.6 and 3.7 in [PS12], respectively. Actually, a slight relaxation of **(H1)** and **(H2)** would be sufficient for the embedding to work, as can be easily seen from their proof: Again, there is a set of clusters $\mathbf{L} \subseteq \mathbf{V}$ such that each cluster in $\mathbf{L}$ contains only vertices of captured degrees at least $(1 + \frac{\eta}{2})k$, there is a matching $M \subseteq \mathbf{G}_{\mathrm{reg}}$, and an edge AB, $A, B \in \mathbf{L}$. One of the following conditions is satisfied

[26] ordinary, in the sense of the classic Regularity Lemma

(H1') the vertices in $A \cup B$ have captured degrees at least $(1+\frac{\eta}{2})k$ into the vertices of $\bigcup(\mathbf{L} \cup V(M))$, or

(H2') the vertices in A have captured degrees at least $(1+\frac{\eta}{2})k$ into the vertices of $\bigcup V(M)$, and the vertices in B have captured degrees at least $(1+\frac{\eta}{2})\frac{k}{2}$ into $\bigcup(\mathbf{L} \cup V(M))$. Further, each edge in M has at most one endvertex in $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$.

It can be seen that Configuration ($\diamond$**10**) is a direct counterpart to **(H1')**.[27] (The counterpart of **(H2')** is contained in Configuration ($\diamond$**9**) and the similarity is somewhat weaker.)

Therefore, we do not include a detailed proof of the embedding procedure in Configuration ($\diamond$**10**), referring the reader to [PS12]. The embedding lemma is formally stated in Lemma 8.25.

8.2 Stochastic process Duplicate(ℓ)

Let us introduce a class of stochastic processes, which we call Duplicate(ℓ) ($\ell \in \mathbb{N}$). These are discrete processes $(X_1, Y_1), (X_2, Y_2), \dots, (X_q, Y_q) \in \{0,1\}^2$ (where $q \in \mathbb{N}$ is arbitrary) satisfying the following.

- For each $i \in [q]$, we have either
  - (a) $X_i = Y_i = 0$ (deterministically), or
  - (b) $X_i = Y_i = 1$ (deterministically), or
  - (c) exactly one of X_i and Y_i is one, and in that case $\mathbf{P}[X_i = 1] = \frac{1}{2}$.
- If the distribution of (X_i, Y_i) is according to (c), then the random choice is made independently of the values (X_j, Y_j) $(j < i)$.
- We have $\sum_{i=1}^{q}(X_i + Y_i) \leqslant \ell$.

Needless to say that this definition is not deep and its purpose is only to adopt the language we shall be using later. The following lemma asserts that the first and second component of a process Duplicate(ℓ) are typically balanced.

Lemma 8.2. *Suppose that* $(X_1, Y_1), (X_2, Y_2), \dots, (X_q, Y_q)$ *is a process in* Duplicate(ℓ). *Then for any* $a > 0$ *we have*

$$\mathbf{P}\left[\sum_{i=1}^{q} X_q - \sum_{i=1}^{q} Y_q \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2\ell}\right) .$$

Proof. We shall be using the following version of the Chernoff bound for sums of independent random variables Z_i, with distribution $\mathbf{P}[Z_i = 1] = \mathbf{P}[Z_i = -1] = \frac{1}{2}$.

$$\mathbf{P}\left[\sum_{i=1}^{n} Z_i \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2n}\right) . \tag{8.1}$$

[27] Observe that some parts of $\mathbf{G}_{\mathrm{reg}}$ are irrelevant in the embedding process of [PS12]. The objects $\mathbf{G}_{\mathrm{reg}}$, $\mathbf{L}$, and M in the structural result of [PS12] correspond to $(\tilde{G}, \mathcal{V})$, $\mathcal{L}^*$, and $\mathcal{M}$ in Configuration ($\diamond$**10**).

Let $J \subseteq [q]$ be the set of all indices i with $X_i + Y_i = 1$. By the definition of $\mathsf{Duplicate}(\ell)$, we have $|J| \leqslant \ell$. By (8.1) we have

$$\mathbf{P}\left[\sum_J (X_i - Y_i) \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2|J|}\right) \leqslant \exp\left(-\frac{a^2}{2\ell}\right) .$$

$\square$

We shall use the stochastic process $\mathsf{Duplicate}$ to guarantee that certain fixed vertex sets do not get overfilled during our tree embedding procedure. $\mathsf{Duplicate}$ is used in Lemmas 8.11 and 8.12 through Lemma 8.10.

8.3 Embedding small trees

When embedding the tree $T_{\triangleright\mathrm{T}1.3}$ in our proof of Theorem 1.3 it will be important to control where different bits of $T_{\triangleright\mathrm{T}1.3}$ go. This motivates the following notation. Let $X_1, \ldots, X_\ell \subseteq V(T)$ be arbitrary vertex sets of a tree T, and let $V_1, \ldots, V_\ell \subseteq V(G)$ be arbitrary vertex sets of a graph G. Then an embedding $\phi : V(T) \to V(G)$ of T in G is an *$(X_1 \hookrightarrow V_1, \ldots, X_\ell \hookrightarrow V_\ell)$-embedding* if $\phi(X_i) \subseteq V_i$ for each $i \in [\ell]$.

We provide several sufficient conditions for embedding a small tree with additional constraints.

The first lemma deals with embedding using an avoiding set.

Lemma 8.3. *Let $\Lambda, k \in \mathbb{N}$ and let $\varepsilon, \gamma \in (0, \frac{1}{2})$ with $\gamma^2 > \varepsilon$. Suppose $\mathfrak{A}$ is a $(\Lambda, \varepsilon, \gamma, k)$-avoiding set with respect to a set $\mathcal{D}$ of $(\gamma k, \gamma)$-dense spots in a graph H. Suppose that $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ are rooted trees with $|\bigcup_i T_i| \leqslant \gamma k/2$. Let $U \subseteq V(H)$ with $|U| \leqslant \Lambda k$, and let $U^* \subseteq \mathfrak{A}$ with $|U^*| \geqslant \varepsilon k + \ell$. Then there are mutually disjoint $(r_i \hookrightarrow U^*, V(T_i) \setminus \{r_i\} \hookrightarrow V(H) \setminus U)$-embeddings of the trees (T_i, r_i) in H.*

Proof. Since $\mathfrak{A}$ is $(\Lambda, \varepsilon, \gamma, k)$-avoiding, there exists a set $Y \subseteq \mathfrak{A}$ with $|Y| \leqslant \varepsilon k$, such that each vertex v in $\mathfrak{A} \setminus Y$ has degree at least γk into some $(\gamma k, \gamma)$-dense spot $D \in \mathcal{D}$ with $|U \cap V(D)| \leqslant \gamma^2 k$. In particular, $U^* \setminus Y$ is large enough so that we can embed all vertices r_i there. We extend this embedding successively to an embedding of $\bigcup_i T_i$, in each step finding a suitable image in $V(D) \setminus U$ for one neighbour of an already embedded vertex $v \in \bigcup_i V(T_i)$. This is possible since the image of v has degree at least $\gamma k - |U \cap V(D)| > \gamma k/2 \geqslant \sum_i v(T_i)$ into $V(D) \setminus U$. $\square$

The next lemma deals with embedding a tree into a nowhere-dense graph, a primal example of which is the graph G_{exp}.

Lemma 8.4. *Let $k \in \mathbb{N}$, let $Q \geqslant 1$ and let $\gamma, \zeta \in (0, 1)$ be such that $128Q\gamma \leqslant \zeta^2$. Let H be a $(\gamma k, \gamma)$-nowhere-dense graph. Let $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ be rooted trees of total order less than $\zeta k/4$. Let $V_1, V_2, U, U^* \subseteq V(H)$ be four sets with $U^* \subseteq V_1$, $|U| < Qk$, $|U^*| > \frac{32Q^2\gamma}{\zeta} k + \ell$, and $\mathrm{deg}^{\min}{}_H(V_j, V_{3-j}) \geqslant \zeta k$ for $j = 1, 2$. Then there are mutually disjoint $(r_i \hookrightarrow U^*, V_{\mathrm{even}}(T_i) \hookrightarrow V_1 \setminus U, V_{\mathrm{odd}}(T_i) \hookrightarrow V_2 \setminus U)$-embeddings of the trees (T_i, r_i) in H.*

Proof. Set $B := \mathbf{shadow}_H(U, \zeta k/2)$. By Fact 7.2, we have $|B| \leqslant \frac{32Q^2\gamma}{\zeta}k \leqslant \frac{\zeta}{4}k$. In particular, $U^* \setminus B$ is large enough to accommodate the images $\phi(r_i)$ of all vertices r_i.

Successively, extend ϕ, in each step mapping a neighbour u of some already embedded vertex $v \in \bigcup_i V(T_i)$ to a yet unused neighbour of $\phi(v)$ in $V_j \setminus (B \cup U)$, where j is either 1 or 2, depending on the parity of $\mathrm{dist}_T(r, v)$. This is possible as $\phi(v)$, lying outside B, has at least $\zeta k/2$ neighbours in $V_i \setminus U$. Thus $\phi(v)$ has at least $\zeta k/4$ neighbours in $V_i \setminus (U \cup B)$, which is more than $\sum_i v(T_i)$. □

The next three standard lemmas deal with embedding trees in a regular or a super-regular pair. We omit their proofs.

Lemma 8.5. *Let $\varepsilon > 0$ and $\beta > 2\varepsilon$. Let (C, D) be an ε-regular pair in a graph H, with $|C| = |D| =: \ell$, and with density $\mathrm{d}(C, D) \geqslant 3\beta$. Suppose that there are sets $X \subseteq C$, $Y \subseteq D$, and $X^* \subseteq X$ satisfying $\min\{|X|, |Y|\} \geqslant 4\frac{\varepsilon}{\beta}\ell$ and $|X^*| > \frac{\beta}{2}\ell$. Let (T, r) be a rooted tree of order $v(T) \leqslant \varepsilon\ell$. Then there exists an $(r \hookrightarrow X^*, V_{\mathrm{even}}(T) \hookrightarrow X, V_{\mathrm{odd}}(T) \hookrightarrow Y)$-embedding of T in H.*

Lemma 8.6. *Let $\beta, \varepsilon > 0$ and $\ell \in \mathbb{N}$ be such that $\beta > 2\varepsilon$. Let (C, D) be an ε-regular pair with $|C| = |D| = \ell$ of density $\mathrm{d}(C, D) \geqslant 3\beta$ in a graph H. Let $(T_1, r_1), (T_2, r_2), \ldots, (T_s, r_s)$ be rooted trees with $v(T_i) \leqslant \varepsilon\ell$ for all $i \in [s]$. Let $U \subseteq V(H)$ fulfill $|C \cap U| = |D \cap U|$, and let $X^* \subseteq (C \cup D) \setminus U$ be such that*

$$|X^*| \geqslant \sum_{i=1}^{s} v(T_i) + 50\beta\ell\ . \tag{8.2}$$

Then there are mutually disjoint $(r_i \hookrightarrow X^*, V(T_i) \hookrightarrow (C \cup D) \setminus U)$-embeddings of the trees (T_i, r_i) in H.

Lemma 8.7. *Let $d > 10\varepsilon > 0$. Suppose that (A, B) forms an (ε, d)-super-regular pair with $|A|, |B| \geqslant \ell$. Let $U_A \subseteq A$, $U_B \subseteq B$ be such that $|U_A| \leqslant |A|/2$ and $|U_B| \leqslant d|B|/4$. Let (T, r) be a rooted tree of order at most $d\ell/4$, and let $v \in A \setminus U_A$ be arbitrary. Then there exists an $(r \hookrightarrow v, V_{\mathrm{even}}(T, r) \hookrightarrow A \setminus U_A, V_{\mathrm{odd}}(T, r) \hookrightarrow B \setminus U_B)$-embedding of T.*

Suppose that we we have a rooted tree (T, r) to be embedded, and its root was already on a vertex $\phi(r)$. Suppose that r has degree $\ell_X + \ell_Y$ in a regular pair (X, Y), where $\ell_X := \deg(\phi(r), X), \ell_Y := \deg(\phi(r), Y)$, with $\ell_X \geqslant \ell_Y$, say. The hope is that we can embed T in (X, Y) as long as $v(T)$ is a bit smaller than $\ell_X + \ell_Y$. For this, the greedy strategy does not work (see Figure 8.5) and we need to be somewhat more careful. We split the embedding process into two stages. In the first stage we choose a subset of the components of $T - r$ of total order approximately $2\min\big(\ell_X, \ell_Y\big) = 2\ell_Y$. When embedding these, we choose orientations of each component in such a way that the image is approximately balanced with respect to X and Y. In the second stage we embed the remaining components so that their roots are embedded in X. We refer to the first stage as embedding in an *balanced way*, and as embedding in an *unbalanced way* to the second stage.

The next lemma says that each regular pair can be filled-up in a balanced way by trees.

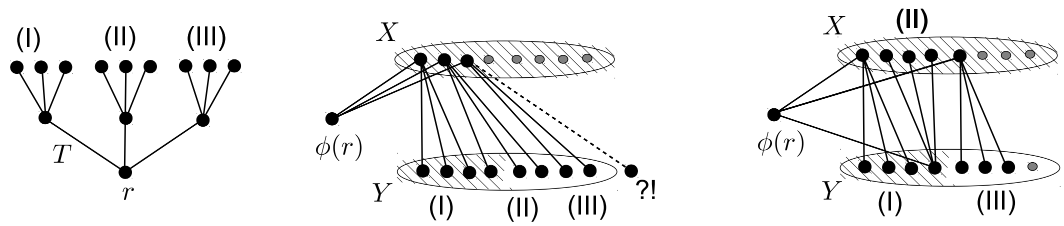

Figure 8.5: An example of a rooted tree (T, r), depicted on the left. The forest $T - r$ has three components (I), (II), (III) of total order 12. Say the vertex r is embedded so that for the regular pair (X, Y) we have $\deg(\phi(r), X) = 8$, $\deg(\phi(r), Y) = 4$ (neighbourhoods of $\phi(r)$ hatched). While the greedy strategy does not work (middle), splitting the process into a balanced and an unbalanced stage (right) does — here the components (I) and (II) are embedded in the balanced stage and the component (III) in the unbalanced stage.

Lemma 8.8. *Let G be a graph, $v \in V(G)$ be a vertex, $\mathcal{M}$ be an $(\varepsilon, d, \nu k)$-semiregular matching in G, and $\{f_{CD}\}_{(C,D)\in\mathcal{M}}$ a family of integers between $-\tau k$ and τk. Suppose (T, r) is a rooted tree,*

$$v(T) \leqslant \left(1 - \frac{4(\varepsilon + \frac{\tau}{\nu})}{d - 2\varepsilon}\right) |V(\mathcal{M})| \ ,$$

with the property that each component of $T - r$ has order at most τk. If $V(\mathcal{M}) \subseteq \mathrm{N}_G(v)$ then there exists an $(r \hookrightarrow v, V(T - r) \hookrightarrow V(\mathcal{M}))$-embedding ϕ of T such that for each $(C, D) \in \mathcal{M}$ we have $|C \cap \phi(T)| + f_{CD} = |D \cap \phi(T)| \pm \tau k$.

The proof of Lemma 8.8 is standard, and is given for example in [HP, Lemma 6.6].

Lemma 8.8 suggests the following definitions. The *discrepancy* of a set X with respect to a pair of sets (C, D) is the number $|C \cap X| - |D \cap X|$. X is *s-balanced* with respect to a semiregular matching $\mathcal{M}$ if the discrepancy of X with respect to each $(C, D) \in \mathcal{M}$ is at most s in absolute value.

Lemma 8.9. *Let G be a graph, $v \in V(G)$ be a vertex, $\mathcal{M}$ be an $(\varepsilon, d, \nu k)$-semiregular matching in G with an $\mathcal{M}$-cover $\mathcal{F}$, and $U \subseteq V(G)$. Suppose (T, r) is a rooted tree with*

$$v(T) + |U| \leqslant \deg_G\left(v, V(\mathcal{M}) \setminus \bigcup \mathcal{F}\right) - \frac{4(\varepsilon + \frac{\tau}{\nu})}{d - 2\varepsilon}|V(\mathcal{M})| \ ,$$

such that each component of $T - r$ has order at most τk. Then there exists an $(r \hookrightarrow v, V(T - r) \hookrightarrow V(\mathcal{M}) \setminus U)$-embedding ϕ of T.

The proof of Lemma 8.9 is again standard and we again omit it.

The following lemma uses a probabilistic technique to embed a shrub while reserving a set of vertices in the host graph for later use. We wish the reserved set to use about as much space inside certain given sets P_i as the image of our shrub does. (In later applications the sets P_i correspond to neighbourhoods of vertices which are still 'active'.)

Lemma 8.10 will find an immediate application in all the remaining lemmas of this subsection. However it is really necessary only for Lemmas 8.11–8.12, which deal with embedding shrubs in the presence of one of the Configurations ($\diamond$**6**)–($\diamond$**8**). For Lemmas 8.13 and 8.14, which are for Configurations ($\diamond$**3**) and ($\diamond$**4**) a simpler auxiliary lemma (without reservations) would suffice.

Lemma 8.10. *Let H be a graph, let $X^*, X_1, X_2, P_1, P_2, \ldots, P_L \subseteq V(H)$, and let $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ be rooted trees, such that $L \leqslant k$, $|P_j| \leqslant k$ for each $j \in [L]$, and $|X^*| \geqslant 2\ell$. Suppose that $\mathrm{deg}^{\min}(X_1 \cup X^*, X_2) \geqslant 2\sum v(T_i)$ and $\mathrm{deg}^{\min}(X_2, X_1) \geqslant 2\sum v(T_i)$.*

Then there exist pairwise disjoint $(r_i \hookrightarrow X^*, V_{\mathrm{even}}(T_i, r_i) \setminus \{r_i\} \hookrightarrow X_1, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow X_2)$-embeddings ϕ_i of T_i in G and a set $C \subseteq (X_1 \cup X_2) \setminus \bigcup \phi_i(T_i)$ of size $\sum v(T_i)$ such that for each $j \in [L]$ we have

$$|P_j \cap \bigcup \phi_i(T_i)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.3}$$

Proof. Let $m := \sum v(T_i)$.

We construct pairwise disjoint random $(r_i \hookrightarrow X^*, V_{\mathrm{even}}(T_i, r_i) \setminus \{r_i\} \hookrightarrow X_1, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow X_2)$-embeddings ϕ_i and a set $C \subseteq V(H) \setminus \bigcup \phi_i(T_i)$ which satisfies (8.3) with positive probability. Then the statement follows.

Enumerate the vertices of $\bigcup T_i$ as $\bigcup V(T_i) = \{v_1, \ldots, v_m\}$ such that $v_i = r_i$ for $i = 1, \ldots, \ell$, and such that for each $j > \ell$ we have that the parent of v_j lies is the set $\{v_1, \ldots, v_{j-1}\}$. Pick pairwise disjoint sets $A_1, \ldots, A_\ell \subseteq X^*$ of size two. Uniformly at random denote one element of A_j as x_j and the other as y_j.

Now, successively for $i = \ell + 1, \ldots, m$, we shall define vertices x_i and y_i. Let r denote the root of the tree in which v_i lies, and let $v_s = \mathrm{Par}(v_i)$. We shall choose $x_i, y_j \in X_{j_i}$ where $j_i = \mathrm{dist}(r, v_i) \bmod 2 + 1$. In step i, proceed as follows. Since $x_s \in X_{j_s}$ (or since $x_s \in X^*$), we have

$$\deg(x_s, X_{j_i} \setminus \bigcup_{h<i} \{x_h, y_h\}) \geqslant 2.$$

Hence, we may take an arbitrary subset $A_i \subseteq (\mathrm{N}(x_s) \cap X_{j_i}) \setminus \bigcup_{h<i} \{x_h, y_h\}$ of size exactly two. As above, randomly label its elements as x_i and y_i independently of all other choices.

The choices of the maps $(v_j \mapsto x_j)_{j=1}^m$ determine $\phi_1, \ldots, \phi_\ell$. Then $C := \{y_1, \ldots, y_m\}$ has size exactly m and avoids $\bigcup \phi_i(T_i)$.

For each $j \in [L]$ we set up a stochastic process $\mathfrak{S}^{(j)} = \left((X_i^{(j)}, Y_i^{(j)})\right)_{i=1}^m$, defined by $X_i^{(j)} = \mathbf{1}_{\{x_i \in P_j\}}$ and $Y_i^{(j)} = \mathbf{1}_{\{y_i \in P_j\}}$. Note that $\mathfrak{S}^{(j)} \in \mathsf{Duplicate}(|P_j|) \subseteq \mathsf{Duplicate}(k)$. Thus, for a fixed $j \in [L]$, by Lemma 8.2, the probability that $|P_j \cap (\bigcup \phi_i(T_i))| > |P_j \cap C| + k^{3/4}$ is at most $\exp(-\sqrt{k}/2)$. Using the union bound over all $j \in [L]$ we get that Property 8.5 holds with probability at least

$$1 - L \cdot \exp\left(-\frac{\sqrt{k}}{2}\right) > 0\ .$$

This finishes the proof. $\square$

We now get to the first application of Lemma 8.10.

Lemma 8.11. *Assume we are in Setting 7.4. Suppose that the sets V_2, V_3 are such that for $j = 2, 3$ we have*

$$\deg^{\min}{}_H(V_j, V_{5-j}) \geqslant \delta k, \tag{8.4}$$

where $\delta > 300/k$, and H is a $(\gamma k, \gamma)$-nowhere dense graph. Suppose that $U, U^*, P_1, P_2, \ldots, P_L \subseteq V(G)$, and $L \leqslant k$, are such that $|U| \leqslant \frac{\delta}{24\sqrt{\gamma}}k$, $U^* \subseteq V_2$, $|U^*| \geqslant \frac{\delta}{4}k$, and $|P_j| \leqslant k$ for each $j \in [L]$. Let (T, r) be a rooted tree of order at most $\delta k/8$.

Then there exists a $(r \hookrightarrow U^*, V_{\mathrm{even}}(T, r) \setminus \{r\} \hookrightarrow V_2 \setminus U, V_{\mathrm{odd}}(T, r) \hookrightarrow V_3 \setminus U)$-embedding ϕ of T in G and a set $C \subseteq (V_2 \cup V_3) \setminus (U \cup \phi(T))$ of size $v(T)$ such that for each $j \in [L]$ we have

$$|P_j \cap \phi(T)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.5}$$

Proof. Set $B := \mathbf{shadow}_{G_{\exp}}(U, \delta k/4)$. By Fact 7.2, we have that $|B| \leqslant 64\frac{\gamma}{\delta}(\frac{\delta}{24\sqrt{\gamma}})^2 k \leqslant \frac{\delta}{4}k - 2$. In particular, $X^* := U^* \setminus B$ has size at least 2. Set $X_1 := V_2 \setminus (U \cup B)$ and set $X_2 := V_3 \setminus (U \cup B)$. Using (8.4), we find that

$$\deg^{\min}{}_{G_{\exp}}(X_j, X_{3-j}) \geqslant \delta k - \deg^{\max}{}_{G_{\exp}}(X_j, U) - |B| \geqslant \delta k - \frac{\delta}{4}k - \frac{\delta}{4}k \geqslant 2v(T)$$

for $j = 1, 2$. We may thus apply Lemma 8.10 to obtain the desired embedding ϕ and the set C. ☐

Lemma 8.12. *Assume Setting 7.4 and Setting 7.7. Suppose that we are given sets $Y_1, Y_2 \subseteq \mathfrak{P}_1 \setminus \bar{V}$ with $Y_1 \subseteq \mathfrak{A}$, and*

(i) $\deg^{\max}{}_{G_{\mathcal{D}}}(Y_1, \mathfrak{P}_1 \setminus Y_2) \leqslant \frac{\eta\gamma}{400}$, *and*

(ii) $\deg^{\min}{}_{G_{\mathcal{D}}}(Y_2, Y_1) \geqslant \delta k$.

Suppose that $U, U^*, P_1, P_2, \ldots, P_L \subseteq V(G)$ are sets such that $|U| \leqslant \frac{\Lambda\delta}{2\Omega^*}k$, $U^* \subseteq Y_1$, with $|U^*| \geqslant \frac{\delta}{4}k$, $|P_j| \leqslant k$ for each $j \in [L]$, and $L \leqslant k$. Suppose $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ are rooted trees of total order at most $\delta k/1000$. Suppose further that $\delta < \eta\gamma/100$, $\varepsilon' < \delta/1000$, and $k > 1000/\delta$.

Then there exist pairwise disjoint $(r_i \hookrightarrow U^*, V_{\mathrm{even}}(T_i, r_i) \hookrightarrow Y_1 \setminus U, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow Y_2 \setminus U)$-embeddings ϕ_i of T_i in G and a set $C \subseteq V(G - \bigcup \phi_i(T_i))$ of size $\sum v(T_i)$ such that for each $j \in [L]$ we have that

$$|P_j \cap \bigcup \phi_i(T_i)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.6}$$

Proof. Set $U' := \mathbf{shadow}_{G_{\mathcal{D}}}(U, \delta k/2) \cup U$. By Fact 7.1, we have $|U'| \leqslant \Lambda k$. As Y_1 is a $(\Lambda, \varepsilon', \gamma, k)$-avoiding set, by Definition 4.6 there exists a set $B \subseteq Y_1$, $|B| \leqslant \varepsilon' k$ such that for all $v \in Y_1 \setminus B$ there exists a dense spot $D_v \in \mathcal{D}$ with $v \in V(D_v)$ and $|V(D_v) \cap U'| \leqslant \gamma^2 k$. As Y_1 is disjoint from $\bar{V}$, by Definition 7.6(4) and by (7.15), we have that $\deg_{D_v}(v, V(D_v)^{\restriction 1}) \geqslant \frac{\eta\gamma}{200}k$. By (i), we have that $\deg_{G_{\mathcal{D}}}(v, V(D_v)^{\restriction 1} \setminus Y_2) < \frac{\eta\gamma}{400}k$, and hence,

$$\deg_{G_{\mathcal{D}}}\left(v, (V(D_v)^{\restriction 1} \cap Y_2) \setminus U'\right) \geqslant \frac{\eta\gamma k}{400} - \gamma^2 k \geqslant \frac{\eta\gamma k}{800}\ .$$

Thus,

$$\deg^{\min}{}_{G_{\mathcal{D}}}(Y_1 \setminus B, Y_2 \setminus (U' \cup B)) \geqslant \frac{\eta\gamma k}{800} - \varepsilon' k \geqslant 2 \sum v(T_i)\ . \tag{8.7}$$

Further, by the definition of U' and by (ii), we have

$$\deg^{\min}{}_{G_{\mathcal{D}}}(Y_2 \setminus U', Y_1 \setminus (U \cup B)) \geqslant \frac{\delta k}{2} - \varepsilon' k \geqslant 2\sum v(T_i) \; . \tag{8.8}$$

Set $X^* := U^* \setminus B$, and note that $|X^*| \geqslant \delta k/4 - \varepsilon' k \geqslant 2\ell$. Set $X_1 := Y_1 \setminus (U \cup B)$ and $X_2 := Y_2 \setminus (U' \cup B)$. Inequalities (8.7) and (8.8) guarantee that we may apply Lemma 8.10 to obtain the desired embeddings ϕ_i. □

Lemma 8.13. *Assume Setting 7.4. Suppose that the sets $L', L'', \mathbf{\Psi'}, \mathbf{\Psi''}, V_1, V_2$ witness Configuration $(\diamond\mathbf{3})(0, 0, \gamma/4, \delta)$. Suppose that $U, U^* \subseteq V(G)$ are sets such that $|U| \leqslant k$, $U^* \subseteq V_1$, $|U^*| \geqslant \frac{\delta}{4}k$. Suppose (T, r) is a rooted tree of order at most $\delta k/1000$. Suppose further that $\delta \leqslant \gamma/100$, $\varepsilon' < \delta/1000$, and $4\Omega^*/\delta \leqslant \Lambda$.*

Then there is an $(r \hookrightarrow U^*, V_{\mathrm{even}}(T, r) \setminus \{r\} \hookrightarrow V_1 \setminus U, V_{\mathrm{odd}}(T, r) \hookrightarrow V_2 \setminus U)$-embedding of T in G.

Proof. The proof of this lemma is very similar to the one of Lemma 8.12 (in fact, even easier). Set $U' := \mathbf{shadow}_{G_{\mathcal{D}}}(U, \delta k/2) \cup U$ and note that $|U'| \leqslant \Lambda k$ by Fact 7.1. As V_1 is $(\Lambda, \varepsilon', \gamma, k)$-avoiding, by Definition 4.6 there is a set $B \subseteq V_1$, $|B| \leqslant \varepsilon' k$ such that for all $v \in V_1 \setminus B$ there exists a dense spot $D_v \in \mathcal{D}$ with $\deg_{D_v}(v, V(D_v) \setminus U') \geqslant \gamma k/2$. By (7.31), we know that $\deg_{G_{\mathcal{D}}}(v, V(D_v) \setminus V_2) \leqslant \gamma k/4$, and hence, $\deg_{G_{\mathcal{D}}}\big(v, (V(D_v) \cap V_2) \setminus U'\big) \geqslant \gamma k/4$. Thus,

$$\deg^{\min}{}_{G_{\mathcal{D}}}(V_1 \setminus B, V_2 \setminus U') \geqslant \frac{\gamma k}{4} \geqslant 2v(T) \; . \tag{8.9}$$

Further, by the definition of U' and by (7.32), we have

$$\deg^{\min}{}_{G_{\mathcal{D}}}(V_2 \setminus U', V_1 \setminus U) \geqslant \frac{\delta k}{2} \geqslant 2(T) \; . \tag{8.10}$$

Set $X^* := U^* \setminus B$, and note that $|X^*| \geqslant \delta k/4 - \varepsilon' k \geqslant 2$. Set $X_1 := V_1 \setminus (U \cup B)$ and $X_2 := V_2 \setminus (U' \cup B)$. Inequalities (8.9) and (8.10) guarantee that we may apply Lemma 8.10 (with empty sets P_i) to obtain the desired embedding ϕ. □

Lemma 8.14. *Assume Setting 7.4. Suppose that the sets $L', L'', \mathbf{\Psi'}, \mathbf{\Psi''}, V_1, \mathfrak{A}', V_2$ witness Configuration $(\diamond\mathbf{4})(0, 0, \gamma/4, \delta)$. Suppose that $U \subseteq V(G)$, $U^* \subseteq V_1$ are sets such that $|U| \leqslant k$ and $|U^*| \geqslant \frac{\delta}{4}k$. Suppose (T, r) is a rooted tree of order at most $\delta k/20$ with a fruit r'. Suppose further that $4\varepsilon' \leqslant \delta \leqslant \gamma/100$, and $\Lambda \geqslant 300(\frac{\Omega^*}{\delta})^3$.*

Then there exists an $(r \hookrightarrow U^*, r' \hookrightarrow V_1 \setminus U, V(T) \setminus \{r, r'\} \hookrightarrow (\mathfrak{A}' \cup V_2) \setminus U)$-embedding of T in G.

Proof. Set

$$U' := \tilde{U} \cup \mathbf{shadow}_{G_\nabla - \mathbf{\Psi}}(U, \delta k/4) \cup \mathbf{shadow}^{(2)}_{G_\nabla - \mathbf{\Psi}}(\tilde{U}, \delta k/4)$$

and let

$$U'' := \tilde{U} \cup \mathbf{shadow}_{G_{\mathcal{D}}}(U', \delta k/2).$$

We use Fact 7.1 to see that $|U'| \leqslant \frac{\delta}{4\Omega^*}\Lambda k$ and $|U''| \leqslant \Lambda k$. We then use Definition 4.6 and (7.36) to find a set $B \subseteq \mathfrak{A}'$ of size at most $\varepsilon' k$ such that

$$\deg^{\min}{}_{G_{\mathcal{D}}}(\mathfrak{A}' \setminus B, V_2 \setminus U'') \geqslant 2v(T) \; . \tag{8.11}$$

Using (8.11), and employing (7.33) and (7.35), we see that we may apply Lemma 8.10 with $X^*_{\triangleright \mathrm{L}8.10} := U^*$, $X_{1,\triangleright \mathrm{L}8.10} := \mathfrak{A}' \setminus (B \cup U')$ and $X_{2,\triangleright \mathrm{L}8.10} := V_2 \setminus U''$ (and with empty sets P_i) in order to embed the tree $T - T(r, \uparrow r')$ rooted at r. Then embed $T(r, \uparrow r')$, by applying Lemma 8.10 a second time, using (7.33) and (7.34). $\square$

8.4 Main embedding lemmas

For this section, we need to introduce the notion of a ghost. Given a semiregular matching $\mathcal{N}$, we call an involution $\mathfrak{d} : V(\mathcal{N}) \to V(\mathcal{N})$ with the property that $\mathfrak{d}(S) = T$ for each $(S, T) \in \mathcal{N}$ a *matching involution*.

Assume Setting 7.4 and fix a matching involution $\mathfrak{b}$ for $\mathcal{M}_A \cup \mathcal{M}_B$. For any set $U \subseteq V(G)$ we then define by

$$\mathbf{ghost}(U) := U \cup \mathfrak{b}\big(U \cap V(\mathcal{M}_A \cup \mathcal{M}_B)\big) .$$

Clearly, we have that $|\mathbf{ghost}(U)| \leqslant 2|U|$, and $|\mathbf{ghost}(U) \cap S| = |\mathbf{ghost}(U) \cap T|$ for each $(S, T) \in \mathcal{M}_A \cup \mathcal{M}_B$.

The notion of ghost extends to other semiregular matchings. If $\mathcal{N}$ is a semiregular matching and $\mathfrak{d}$ a matching involution for $\mathcal{N}$ then we write $\mathbf{ghost}_{\mathfrak{d}}(U) := U \cup \mathfrak{d}\big(U \cap V(\mathcal{N})\big)$.

8.4.1 Embedding in Configuration ($\diamond\mathbf{1}$)

This subsection contains an easy observation that $\mathbf{trees}(k) \subseteq G$ in case G contains Configuration ($\diamond\mathbf{1}$).

Lemma 8.15. *Let G be a graph, and let $A, B \subseteq V(G)$ be such that* $\deg^{\min}(G[A, B]) \geqslant k/2$*, and* $\deg^{\min}(A) \geqslant k$*. Then* $\mathbf{trees}(k) \subseteq G$*.*

Proof. Let $T \in \mathbf{trees}(k)$ have colour classes X and Y, with $|X| \geqslant k/2 \geqslant |Y|$. By Fact 2.1, for the set W of those leaves of T that lie in X, we have $|X \setminus W| \leqslant k/2$. We embed $T - W$ greedily in G, mapping Y to A and $X \setminus W$ to B. We then embed W using the fact that $\deg^{\min}(A) \geqslant k$. $\square$

8.4.2 Embedding in Configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$)

In this section we show how to embed $T_{\triangleright \mathrm{T}1.3}$ in the presence of configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$). As outlined in Section 8.1.1 our main embedding lemma, Lemma 8.18, builds on Lemma 8.17 which handles Stage 1 of the embedding, and Lemma 8.16 which handles Stage 2.

Lemma 8.16. *Assume we are in Setting 7.4. Suppose L'', L' and $\mathbf{\Psi}'$ witness Preconfiguration* $(\clubsuit)(\frac{10^5\Omega^*}{\eta})$*. Let (T, r) be a rooted tree of order at most $\gamma^2\nu k/6$. Let $U \subseteq V(G)$ with $|U| + v(T) \leqslant k$, and let $v \in \mathbf{\Psi}' \setminus U$. Then there exists an $(r \hookrightarrow v, V(T) \hookrightarrow V(G) \setminus U)$-embedding of (T, r).*

Proof. We proceed by induction on the order of T. The base $v(T) \leqslant 2$ obviously holds. Let us assume Lemma 8.16 is true for all trees T' with $v(T') < v(T)$.

Let $U_1 := \mathbf{shadow}_{G_\nabla}(U - \mathbf{\Psi}, \eta k/200)$, and $U_2 := \bigcup\{C \in \mathbf{V} \ : \ |C \cap U| \geqslant \frac{1}{2}|C|\}$. We have $|U_1| \leqslant \frac{200\Omega^*}{\eta}k$ by Fact 7.1, and $|U_2| \leqslant 2|U|$. Set

$$L_{\mathfrak{A}} := L'' \cap \mathbf{shadow}_{G_\nabla}(\mathfrak{A}, \frac{\eta k}{50}),$$
$$L_{\mathbf{\Psi}} := L'' \cap \mathbf{shadow}_{G_\nabla}\left(\mathbf{\Psi}, |U \cap \mathbf{\Psi}| + \frac{\eta k}{50}\right), \text{ and}$$
$$L_{\mathbf{V}} := L'' \cap \mathbf{shadow}_{G_{\mathrm{reg}}}\left(V(G_{\mathrm{reg}}), (1 + \frac{\eta}{50})k - |U \cap \mathbf{\Psi}|\right) .$$

Observe that $L_{\mathbf{V}} \subseteq \bigcup \mathbf{V}$ and that since $L'' \subseteq \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus \mathbf{\Psi}$, we have

$$L'' \subseteq V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup L_{\mathbf{\Psi}} \cup L_{\mathfrak{A}} \cup L_{\mathbf{V}} .$$

As by (7.30), we have $\deg_G(v, L'') \geqslant \frac{10^5\Omega^* k}{\eta} > 5(|U \cup U_1 \cup U_2| + v(T) + \eta k)$, one of the following five cases must occur.

Case I: $\deg_G(v, V(G_{\mathrm{exp}}) \setminus U) > v(T) + \eta k$. Lemma 8.4 gives an embedding of the forest $T - r$ (whose components are rooted at neighbours of r). The input sets/parameters of Lemma 8.4 are $Q_{\triangleright \mathrm{L}8.4} := 1$, $\zeta_{\triangleright \mathrm{L}8.4} := 12\sqrt{\gamma}$, $U^*_{\triangleright \mathrm{L}8.4} := (\mathrm{N}_G(v) \cap V(G_{\mathrm{exp}})) \setminus U$, $U_{\triangleright \mathrm{L}8.4} := U$, $V_1 = V_2 := V(G_{\mathrm{exp}})$.

Case II: $\deg_G(v, \mathfrak{A} \setminus U) > v(T) + \eta k$. Lemma 8.3 gives an embedding of the forest $T - r$ (whose components are rooted at neighbours of r). The input sets/parameters of Lemma 8.3 are $U^*_{\triangleright \mathrm{L}8.3} := (\mathrm{N}_G(v) \cap \mathfrak{A}) \setminus U$, $U_{\triangleright \mathrm{L}8.3} := U$ and $\varepsilon_{\triangleright \mathrm{L}8.3} := \varepsilon' \leqslant \eta$. Here, and below, we tacitly implicitly assume parameters of the same name to be the same, i.e. $\gamma_{\triangleright \mathrm{L}8.3} := \gamma$.

Case III: $\deg_G(v, L_{\mathfrak{A}} \setminus (U \cup U_1)) > v(T) + \eta k$. We only outline the strategy. Embed the children of r in $L_{\mathfrak{A}} \setminus (U \cup U_1)$ using a map $\phi : \mathrm{Ch}_T(r) \to L_{\mathfrak{A}} \setminus (U \cup U_1)$. By definition of $L_{\mathfrak{A}}$, and U_1, we have $\deg_{G_\nabla}(\phi(w), \mathfrak{A} \setminus U) > \frac{\eta k}{100}$ for each $w \in \mathrm{Ch}_T(r)$. Now, for every $w \in \mathrm{Ch}_T(r)$ we can proceed as in Case II to extend this embedding to the rooted tree $\big(T(r, \uparrow w), w\big)$. That is, Case III is "Case II with an extra step in the beginning".

Case IV: $\deg_G(v, L_{\mathbf{\Psi}} \setminus U) > v(T) + \eta k$. We embed the children $\mathrm{Ch}_T(r)$ of r in distinct vertices of $L_{\mathbf{\Psi}} \setminus U$. This is possible by the assumption of Case IV.

Now, (7.28) implies that $\mathrm{deg^{min}}_{G_\nabla}(L_{\mathbf{\Psi}}, \mathbf{\Psi'}) \geqslant |U \cap \mathbf{\Psi}| + \frac{\eta k}{100}$. Consequently, $\mathrm{deg^{min}}_{G_\nabla}(L_{\mathbf{\Psi}}, \mathbf{\Psi'} \setminus U) \geqslant \frac{\eta k}{100}$. Therefore, for each $w \in \mathrm{Ch}_T(r)$ embedded in $L_{\mathbf{\Psi}} \setminus U$ we can find an embedding of $\mathrm{Ch}_T(w)$ in $\mathbf{\Psi'} \setminus U$ such that the images of grandchildren of r are disjoint. We fix such an embedding. We can now apply induction. More specifically, for each grandchild u of r we embed the rooted tree $\big(T(r, \uparrow u), u\big)$ using Lemma 8.16 (employing induction) using the updated set U, to which the images of the newly embedded vertices were added.

Case V: $\deg_G(v, L_{\mathbf{V}} \setminus (U \cup \cup U_1 \cup U_2)) \geqslant v(T)$. Let $u_1, \ldots, u_\ell$ be the children of r. Let us consider arbitrary distinct neighbours $x_1, \ldots, x_\ell \in L_{\mathbf{V}} \setminus (U \cup U_1 \cup U_2)$ of v. Let $T_i := T(r, \uparrow u_i)$. We sequentially embed the rooted trees (T_i, u_i), $i = 1, \ldots, \ell$, writing ϕ for the embedding. In step i, consider the set $W_i := \left(U \cup \bigcup_{j<i} \phi(T_j)\right) \setminus \mathbf{\Psi}$. Let $D_i \in \mathbf{V}$ be the cluster containing x_i. By definition of $L_{\mathbf{V}}$ and of U_1,

$$\deg_{G_{\mathrm{reg}}}(x_i, V(G_{\mathrm{reg}}) \setminus W_i) \geqslant \frac{\eta k}{50} - \frac{\eta k}{200} \geqslant \frac{\eta k}{100} .$$

Fact 4.11 yields a cluster $C_i \in \mathbf{V}$ such that

$$\deg_{G_{\mathrm{reg}}}(x_i, C_i \setminus W_i) \geqslant \frac{\eta}{100} \cdot \frac{\gamma \mathfrak{c}}{2(\Omega^*)^2} > \frac{\gamma^2 \mathfrak{c}}{2} + v(T) > \frac{12\varepsilon' \mathfrak{c}}{\gamma^2} + v(T) \ .$$

In particular there is at least one edge from $E(G_{\mathrm{reg}})$ between C_i and D_i, and therefore, (C_i, D_i) forms an ε'-regular pair of density at least γ^2 in G_{reg}. Map u_i to x_i and let $F_1, \ldots, F_m$ be the components of the forest $T_i - u_i$. We now sequentially embed the trees F_j in the pair (D_i, C_i) using Lemma 8.5, with $X_{\triangleright\mathrm{L}8.5} := C_i \setminus (W_i \cup \bigcup_{q<j} \phi(F_q))$, $X^*_{\triangleright\mathrm{L}8.5} := \mathrm{N}_{G_{\mathrm{reg}}}(x_i, X_{\triangleright\mathrm{L}8.5})$, $Y_{\triangleright\mathrm{L}8.5} := D_i \setminus (W_i \cup \{x_i\} \cup \bigcup_{q<j} \phi(F_q))$, $\varepsilon_{\triangleright\mathrm{L}8.5} := \varepsilon'$, and $\beta_{\triangleright\mathrm{L}8.5} := \gamma^2/3$. $\square$

We are now ready for the lemma that will handle Stage 1 in configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$.

Lemma 8.17. *Assume we are in Setting 7.4, with $L'', L', \mathbf{\Psi}'$ witnessing $(\clubsuit)(\Omega^\star)$ in G. Let $U \subseteq V(G) \setminus \mathbf{\Psi}$ and let (T, r) be a rooted tree with $v(T) \leqslant k/2$ and $|U| + v(T) \leqslant k$. Suppose that each component of $T - r$ has order at most τk. Let $x \in (L'' \cap \mathbb{YB}) \setminus \bigcup_{i=0}^{2} \mathbf{shadow}^{(i)}_{G_\nabla}(\mathbf{ghost}(U), \eta k/1000)$.*

Then there is a subtree T' of T with $r \in V(T')$ which has an $(r \hookrightarrow x, V(T') \setminus \{r\} \hookrightarrow V(G) \setminus \mathbf{\Psi})$-embedding ϕ. Further, the components of $T - T'$ can be partitioned into two (possibly empty) sets $\mathcal{C}_1$, $\mathcal{C}_2$, such that the following two assertions hold.

(a) If $\mathcal{C}_1 \neq \emptyset$, then $\mathrm{deg^{min}}_{G_\nabla}(\phi(\mathrm{Par}(V(\bigcup \mathcal{C}_1))), \mathbf{\Psi}') > k + \frac{\eta k}{100} - v(T')$,

(b) $\mathrm{Par}(V(\bigcup \mathcal{C}_2)) \subseteq \{r\}$, *and* $\deg_{G_\nabla}(x, \mathbf{\Psi}') > \frac{k}{2} + \frac{\eta k}{100} - v(T' \cup \bigcup \mathcal{C}_1)$.

Proof. Let $\mathcal{C}$ be the set of all components of $T - r$. We start by defining $\mathcal{C}_2$. Then, we have to distribute $T - \bigcup \mathcal{C}_2$ between T' and $\mathcal{C}_1$. First, we find a set $\mathcal{C}_M \subseteq \mathcal{C} \setminus \mathcal{C}_2$ which fits into the matching $\mathcal{M}_A \cup \mathcal{M}_B$ (and thus will form part of T'). Then, we consider the remaining components of $\mathcal{C} \setminus \mathcal{C}_2$: some of these will be embedded entirely, of others we only embed the root, and leave the rest for $\mathcal{C}_1$. Everything embedded will become a part of T'.

Throughout the proof we write **shadow** for $\mathbf{shadow}_{G_\nabla}$.

Set $\overline{V_{\mathrm{good}}} := V_{\mathrm{good}} \setminus \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})$, and choose $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ such that

$$\deg_{G_\nabla}(x, \overline{V_{\mathrm{good}}}) - \frac{\eta k}{30} \ < \ \sum_{S \in \tilde{\mathcal{C}}} v(S) \ \leqslant \ \max\left\{0, \deg_{G_\nabla}\left(x, \overline{V_{\mathrm{good}}}\right) - \frac{\eta k}{40}\right\}. \tag{8.12}$$

Set $\mathcal{C}_2 := \mathcal{C} \setminus \tilde{\mathcal{C}}$. Note that this choice clearly satisfies the first part of (b). Let us now verify the

second part of (b). For this, we calculate

$$\begin{aligned}
\deg_{G_\nabla}(x,\mathbf{\Psi}') &\geqslant \deg_{G_\nabla}(x,V_+\setminus L_\#)-\deg_{G_\nabla}(x,\mathbf{shadow}(\mathbf{ghost}(U),\frac{\eta k}{1000}))\\
&\quad-\deg_{G_\nabla}(x,V_+\setminus(L_\#\cup\mathbf{shadow}(\mathbf{ghost}(U),\frac{\eta k}{1000})\cup\mathbf{\Psi}))\\
&\quad-\deg_{G_\nabla}(x,\mathbf{\Psi}\setminus\mathbf{\Psi}')\\
\text{\tiny (by (7.11), } x\notin\mathbf{shadow}^{(2)}(\mathbf{ghost}(U),\tfrac{\eta k}{1000}),\text{ (8.12), (7.28))}\quad &\geqslant\left(\frac{k}{2}+\frac{\eta k}{20}\right)-\frac{\eta k}{1000}-\left(\sum_{S\in\tilde{\mathcal{C}}}v(S)+\frac{\eta k}{30}\right)-\frac{\eta k}{100}\\
&>\ \frac{k}{2}-\sum_{S\in\tilde{\mathcal{C}}}v(S)+\frac{\eta k}{20}\\
&\geqslant\frac{k}{2}-v(T'\cup\bigcup\mathcal{C}_1)+\frac{\eta k}{100},
\end{aligned}$$

as desired for (b).

Now, set

$$\mathcal{M}:=\left\{(X_1,X_2)\in\mathcal{M}_A\cup\mathcal{M}_B\ :\ \deg_{G_\mathcal{D}}(x,(X_1\cup X_2)\setminus\mathfrak{A})>0\right\}. \tag{8.13}$$

Claim 8.17.1. We have $|V(\mathcal{M})|\leqslant\frac{4(\Omega^*)^2}{\gamma^2}k$.

Proof of Claim 8.17.1. Indeed, let $(X_1,X_2)\in\mathcal{M}$, i.e. $(X_1,X_2)\in\mathcal{M}_A\cup\mathcal{M}_B$ with $\deg_{G_\mathcal{D}}(x,(X_1\cup X_2)\setminus\mathfrak{A})>0$. Then, using Property 4 of Setting 7.4, we see that there exists a cluster $C_{(X_1,X_2)}\in\mathbf{V}$ such that $\deg_{G_\mathcal{D}}(x,C_{(X_1,X_2)})>0$, and either $X_1\subseteq C_{(X_1,X_2)}$ or $X_2\subseteq C_{(X_1,X_2)}$. In particular, there exists a dense spot $(A_{(X_1,X_2)},B_{(X_1,X_2)};F_{(X_1,X_2)})\in\mathcal{D}$ such that $x\in A_{(X_1,X_2)}$, and $X_1\subseteq B_{(X_1,X_2)}$ or $X_2\subseteq B_{(X_1,X_2)}$. By Fact 4.4, there are at most $\frac{\Omega^*}{\gamma}$ such dense spots, let Z denote the union of all vertices contained in these spots. Fact 4.3 implies that $|Z|\leqslant\frac{2(\Omega^*)^2}{\gamma^2}k$. Thus $|V(\mathcal{M})|\leqslant 2|V(\mathcal{M})\cap Z|\leqslant 2|Z|\leqslant\frac{4(\Omega^*)^2}{\gamma^2}k$. □

First we shall embed as many components from $\tilde{\mathcal{C}}$ as possible in $\mathcal{M}$. To this end, consider an inclusion-maximal subset $\mathcal{C}_M$ of $\tilde{\mathcal{C}}$ with

$$\sum_{S\in\mathcal{C}_M}v(S)\leqslant\deg_{G_\nabla}(x,V(\mathcal{M}))-\frac{\eta k}{1000}. \tag{8.14}$$

We aim to utilize the degree of x to $V(\mathcal{M})$ to embed $\mathcal{C}_M$ in $V(\mathcal{M})$ using the regularity method.

Remark 8.17.2. This remark (which may as well be skipped at a first reading) is aimed at those readers that are wondering about a seeming inconsistency of the defining formulas (8.13) for $\mathcal{M}$, and (8.14) for $\mathcal{C}_M$. That is, (8.13) involves the degree in $G_\mathcal{D}$ and excludes the set $\mathfrak{A}$, while (8.14) involves the degree in G_∇. The setting in (8.13) was chosen so that it allows us to control the size of $\mathcal{M}$ in Claim 8.17.1, crucially relying on Property 4 of Setting 7.4. Such a control is necessary to make the regularity method work. Indeed, in each regular pair there may be a small number

of atypical vertices[28], and we must avoid these vertices when embedding the components by the regularity method. Thus without the control on $|\mathcal{M}|$ it might happen that the degree of x is unusable because x sees very small numbers of atypical vertices in an enormous number of sets corresponding to $\mathcal{M}$-vertices. On the other hand, the edges x sends to $\mathfrak{A}$ can be utilized by other techniques in later stages. Once we have defined $\mathcal{M}$ we want to use the full degree to $V(\mathcal{M})$ to ensure we can embed the shrubs as balanced as possible into the $\mathcal{M}$-edges. This is necessary as otherwise part of the degree of x might be unusable for embedding, e.g. because it might go to $\mathcal{M}$-vertices whose partners are already full.

For each $(C,D) \in \mathcal{M}$ we choose $\mathcal{C}_{CD} \subseteq \mathcal{C}_M$ maximal such that

$$\sum_{S \in \mathcal{C}_{CD}} v(S) \leqslant \deg_{G_\nabla}(x, (C \cup D) \setminus \mathbf{ghost}(U)) - (\frac{\gamma}{\Omega^*})^3 |C| , \tag{8.15}$$

and further, we require $\mathcal{C}_{CD}$ to be disjoint from families $\mathcal{C}_{C'D'}$ defined in previous steps. We claim that $\{\mathcal{C}_{CD}\}_{(C,D)\in\mathcal{M}}$ forms a partition of $\mathcal{C}_M$, i.e., all the elements of $\mathcal{C}_M$ are used. Indeed, otherwise, by the maximality of $\mathcal{C}_{CD}$ and since the components of $T - r$ have size at most τk, we obtain

$$\begin{aligned} \sum_{S \in \mathcal{C}_{CD}} v(S) &\geqslant \deg_{G_\nabla}(x, (C \cup D) \setminus \mathbf{ghost}(U)) - (\frac{\gamma}{\Omega^*})^3 |C| - \tau k \\ &\overset{(7.3)}{\geqslant} \deg_{G_\nabla}(x, (C \cup D) \setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3 |C| , \end{aligned} \tag{8.16}$$

for each $(C,D) \in \mathcal{M}$. Then we have

$$\begin{aligned} \sum_{S \in \mathcal{C}_M} v(S) &> \sum_{(C,D)\in\mathcal{M}} \sum_{S \in \mathcal{C}_{CD}} v(S) \\ \text{(by (8.16))} \quad &\geqslant \sum_{(C,D)\in\mathcal{M}} \big(\deg_{G_\nabla}(x, (C \cup D) \setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3 |C| \big) \\ \text{(by Claim 8.17.1 and Fact 5.5)} \quad &\geqslant \deg_{G_\nabla}(x, V(\mathcal{M}) \setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3 \cdot \frac{2(\Omega^*)^2}{\gamma^2} k \\ \text{(as } x \notin \mathbf{shadow}(\mathbf{ghost}(U))) \quad &\geqslant \deg_{G_\nabla}(x, V(\mathcal{M})) - \frac{\eta k}{1000} \\ \text{(by (8.14))} \quad &\geqslant \sum_{S \in \mathcal{C}_M} v(S) , \end{aligned}$$

a contradiction.

We use Lemma 8.6 to embed the components of $\mathcal{C}_{CD}$ in $(C \cup D) \setminus \mathbf{ghost}(U)$ with the following setting: $C_{\triangleright L8.6} := C$, $D_{\triangleright L8.6} := D$, $U_{\triangleright L8.6} := \mathbf{ghost}(U)$, $X^*_{\triangleright L8.6} := (\mathrm{N}_{G_\nabla}(x) \cap (C \cup D)) \setminus U_{\triangleright L8.6}$, and (T_i, r_i) are the rooted trees from $\mathcal{C}_{CD}$ with the roots being the neighbours of r. The constants in Lemma 8.6 are $\varepsilon_{\triangleright L8.6} := \varepsilon'$, $\beta_{\triangleright L8.6} := \sqrt{\varepsilon'}$, and $\ell_{\triangleright L8.6} := |C| \geqslant \nu\pi k$. The rooted trees in $\mathcal{C}_{CD}$ are smaller than $\varepsilon_{\triangleright L8.6}\ell_{\triangleright L8.6}$ by (7.3). Condition (8.2) is satisfied by (8.15), and since $(\gamma/\Omega^*)^3 \geqslant 50\sqrt{\varepsilon'}$.

[28]The issue of atypicality itself could be avoided by preprocessing each pair (S,T) of $\mathcal{M}_A \cup \mathcal{M}_B$ and making it super-regular. However this is not possible for atypicality with respect to a given (but unknown in advance) subpair (S', T').

It remains to deal with the components $\tilde{\mathcal{C}} \setminus \mathcal{C}_M$. In the sequel we shall assume that $\tilde{\mathcal{C}} \setminus \mathcal{C}_M \neq \emptyset$ (otherwise skip this step and go directly to the definition of T' and $\mathcal{C}_1$, with $p = 0$). Thus, by our choice of $\mathcal{C}_M$, we have

$$\sum_{S \in \mathcal{C}_M} v(S) \geqslant \deg_{G_\nabla}(x, V(\mathcal{M})) - \frac{\eta k}{900} \, . \tag{8.17}$$

Let $T_1, T_2, \ldots, T_p$ be the trees of $\tilde{\mathcal{C}} \setminus \mathcal{C}_M$ rooted at the vertices $r_i \in \mathrm{Ch}(r) \cap V(T_i)$. We shall sequentially extend our embedding of $\mathcal{C}_M$ to subtrees $T'_i \subseteq T_i$. Let $U_i \subseteq V(G)$ be the union of the images of $\bigcup \mathcal{C}_M \cup \{r\}$ and of $T'_1, \ldots, T'_i$ under this embedding.

Suppose that we have embedded the trees $T'_1, \ldots, T'_i$ for some $i = 0, 1, \ldots, p-1$. We claim that at least one of the following holds.

(V1) $\deg_{G_\nabla}(x, V(G_{\mathrm{exp}}) \setminus (U \cup U_i)) \geqslant \frac{\eta k}{1000}$,

(V2) $\deg_{G_\nabla}(x, \mathfrak{A} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{1000}$, or

(V3) $\deg_{G_\nabla}(x, L' \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))) \geqslant \frac{\eta k}{1000}$.

Indeed, suppose that none of **(V1)**–**(V3)** holds. Then, first note that since $U \subseteq \mathbf{ghost}(U)$ and since $x \notin \mathbf{shadow}(\mathbf{ghost}(U), \eta k/1000)$, we have

$$\deg_{G_\nabla}(x, U) \leqslant \eta k/1000. \tag{8.18}$$

Also,

$$\deg_{G_\mathcal{D}}(x, V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \deg_{G_\mathcal{D}}(x, V(\mathcal{M}) \cup \mathfrak{A}). \tag{8.19}$$

Thus,

$$\begin{aligned}
&\deg_{G_\nabla}\left(x, V_{\mathrm{good}} \setminus \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})\right) \\
&\text{(by (8.18) and (8.19), def of } V_{\mathrm{good}}\text{)} \quad \leqslant \deg_{G_\nabla}\left(x, \left(V(\mathcal{M}) \cup V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup L'\right) \setminus (U \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})\right) \\
&\qquad + \deg_{G_\nabla}\left(x, \mathbb{L}_{\frac{9}{10}\eta, k}(G_\nabla) \setminus (\mathbf{\Psi} \cup L')\right) + \frac{\eta k}{1000} \\
&\text{(by (7.30))} \quad \leqslant \deg_{G_\nabla}\left(x, \left(V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup L'\right) \setminus (V(\mathcal{M}) \cup U \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))\right) \\
&\qquad + \deg_{G_\nabla}(x, V(\mathcal{M})) + \frac{\eta k}{100} + \frac{\eta k}{1000} \\
&\text{(by } \neg\textbf{(V1)}, \neg\textbf{(V2)}, \neg\textbf{(V3)}, \text{ by (8.17))} \quad \leqslant 3 \cdot \frac{\eta k}{1000} + \sum_{j=1}^{i} v(T'_j) + \sum_{S \in \mathcal{C}_M} v(S) + \frac{\eta k}{900} + \frac{\eta k}{100} + \frac{\eta k}{1000} \\
&\qquad < \sum_{S \in \tilde{\mathcal{C}}} v(S) + \frac{\eta k}{40} \, ,
\end{aligned}$$

a contradiction to (8.12).

In cases **(V1)**–**(V2)** we shall embed the entire tree $T'_{i+1} := T_{i+1}$. In case **(V3)** we either embed the entire tree $T'_{i+1} := T_{i+1}$, or embed only one vertex $T'_{i+1} := r_{i+1}$ (that will only happen in case **(V3c)**). In the latter case, we keep track of the components of $T_{i+1} - r_{i+1}$ in the set $\mathcal{C}_{1,i+1}$ (we tacitly assume we set $\mathcal{C}_{1,i+1} := \emptyset$ in all cases other than **(V3c)**). The union of the sets $\mathcal{C}_{1,i}$ will later form the set $\mathcal{C}_1$. Let us go through our three cases in detail.

In case **(V1)** we embed T_{i+1} rooted at r_{i+1} using Lemma 8.4 for one tree (i.e. $\ell_{\triangleright \mathrm{L8.4}} := 1$) with the following sets/parameters: $H_{\triangleright \mathrm{L8.4}} := G_{\exp}$, $U_{\triangleright \mathrm{L8.4}} := U \cup U_i$, $U^*_{\triangleright \mathrm{L8.4}} := \mathrm{N}_{G_\nabla}(x) \cap (V(G_{\exp}) \setminus (U \cup U_i))$, $V_1 = V_2 := V(G_{\exp})$, $Q_{\triangleright \mathrm{L8.4}} := 1$, $\zeta_{\triangleright \mathrm{L8.4}} := \rho$, and $\gamma_{\triangleright \mathrm{L8.4}} := \gamma$. Note that $|U \cup U_i| < k$, that $|\mathrm{N}_{G_\nabla}(x) \cap (V(G_{\exp}) \setminus (U \cup U_i))| \geqslant \eta k/1000 > 32\gamma k/\rho + 1$, that $v(T_{i+1}) \leqslant \tau k < \rho k/4$ and that $128\gamma < \rho^2$.

In case **(V2)** we embed T_{i+1} rooted at r_{i+1} using Lemma 8.3 for one tree (i.e. $\ell_{\triangleright \mathrm{L8.3}} := 1$) with the following setting: $H_{\triangleright \mathrm{L8.3}} := G - \mathbf{\Psi}$, $\mathfrak{A}_{\triangleright \mathrm{L8.3}} := \mathfrak{A}$, $U_{\triangleright \mathrm{L8.3}} := U \cup U_i$, $U^*_{\triangleright \mathrm{L8.3}} := \mathrm{N}_{G_\nabla}(x) \cap (\mathfrak{A} \setminus (U \cup U_i))$, $\Lambda_{\triangleright \mathrm{L8.3}} := \Lambda$, $\gamma_{\triangleright \mathrm{L8.3}} := \gamma$, $\varepsilon_{\triangleright \mathrm{L8.3}} := \varepsilon'$. Note that $|U \cup U_i| \leqslant k < \Lambda k$, that $|\mathrm{N}_{G_\nabla}(x) \cap (\mathfrak{A} \setminus (U \cup U_i))| \geqslant \eta k/1000 > 2\varepsilon' k$, and that $v(T_{i+1}) \leqslant \tau k < \gamma k/2$.

We commence case **(V3)** with an auxiliary claim.

Claim 8.17.3. There exists $C_0 \in \mathbf{V}$ such that

$$\deg_{G_{\mathcal{D}}}\left(x, (C_0 \cap L') \setminus (V(G_{\exp}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))\right) \geqslant \frac{\varepsilon'}{\gamma^2}\mathfrak{c} .$$

Proof of Claim 8.17.3. Observe that $L' \setminus (V(G_{\exp}) \cup \mathfrak{A} \cup \mathbf{\Psi} \cup U \cup U_i) \subseteq \bigcup \mathbf{V}$ and that (since $x \in \bigcup \mathbf{V}$)

$$E_{G_\nabla}[x, L' \setminus (V(G_{\exp}) \cup \mathfrak{A} \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))] \subseteq E(G_{\mathcal{D}}) .$$

By Fact 4.11, there are at most $\frac{2(\Omega^*)^2 k}{\gamma^2 \mathfrak{c}}$ clusters $C \in \mathbf{V}$ such that $\deg_{G_{\mathcal{D}}}(x, C) > 0$. Using the assumption **(V3)**, there exists a cluster $C_0 \in \mathbf{V}$ such that

$$\begin{aligned}\deg_{G_{\mathcal{D}}}\left(x, (C_0 \cap L') \setminus (V(G_{\exp}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))\right) &\geqslant \frac{\eta k}{1000} \cdot \frac{\gamma^2 \mathfrak{c}}{2(\Omega^*)^2 k} \\ &\overset{(7.3)}{\geqslant} \frac{\varepsilon'}{\gamma^2}\mathfrak{c} ,\end{aligned}$$

as desired. □

Let us take a cluster C_0 from Claim 8.17.3. We embed the root r_{i+1} of T_{i+1} in an arbitrary neighbour y of x in $(C_0 \cap L') \setminus (V(G_{\exp}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))$.

Let $H \subseteq G$ be the subgraph of G consisting of all edges in dense spots $\mathcal{D}$, and all edges incident with $\mathbf{\Psi}'$. As by (7.28), y has at most $\eta k/100$ neighbours in $\mathbf{\Psi} \setminus \mathbf{\Psi}'$, and since $y \in L' \subseteq \mathbb{L}_{9\eta/10,k}(G_\nabla)$ and $y \notin \mathbf{shadow}(U, \frac{\eta k}{100})$, we find that

$$\begin{aligned}\deg_H\left(y, V(G) \setminus ((U \cup U_i) \cup (\mathbf{\Psi} \setminus \mathbf{\Psi}'))\right) &\geqslant \left(1 + \frac{9\eta}{10}\right)k - \frac{\eta k}{1000} - |U_i| - \frac{\eta k}{100} \\ &> k - |U_i| + \frac{\eta k}{2} .\end{aligned}$$

Therefore, one of the three following subcases must occur. (Recall that $y \notin \mathfrak{A}$ as $y \in C_0 \in \mathbf{V}$.)

(V3a) $\deg_{G_\nabla}(y, \mathfrak{A} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6}$,

(V3b) $\deg_{G_{\mathrm{reg}}}(y, \bigcup \mathbf{V} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6}$, or

(V3c) $\deg_{G_\nabla}(y, \mathbf{\Psi'}) \geqslant k - |U_i| + \frac{\eta k}{6}$.

In case **(V3a)** we embed the components of $T_{i+1} - r_{i+1}$ (as trees rooted at the children of r_{i+1}) using the same technique as in case **(V2)**, with Lemma 8.3.

In **(V3b)** we embed the components of $T_{i+1} - r_{i+1}$ (as trees rooted at the children of r_{i+1}). By Fact 4.11 there exists a cluster $D \in \mathbf{V}$ such that

$$\deg_{G_{\mathrm{reg}}}(y, D \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6} \cdot \frac{\gamma^2 \mathfrak{c}}{2(\Omega^*)^2 k} > \frac{\gamma^2}{2}\mathfrak{c}. \tag{8.20}$$

We use Lemma 8.5 with input $\varepsilon_{\triangleright \mathrm{L}8.5} := \varepsilon'$, $\beta_{\triangleright \mathrm{L}8.5} := \gamma^2$, $C_{\triangleright \mathrm{L}8.5} := D$, $D_{\triangleright \mathrm{L}8.5} := C_0$, $X^*_{\triangleright \mathrm{L}8.5} = X_{\triangleright \mathrm{L}8.5} := D \setminus (U \cup U_i)$ and $Y_{\triangleright \mathrm{L}8.5} := C_0 \setminus (U \cup U_i \cup \{y\})$ to embed the tree T_{i+1} into the pair (C_0, D), by embedding the components of $T_{i+1} - r_{i+1}$ one after the other. The numerical conditions of Lemma 8.5 hold because of Claim (8.17.3) and because of (8.20).

In case **(V3c)** we set $T'_{i+1} := r_{i+1}$ and define $\mathcal{C}_{1,i+1}$ as set of all components of $T_{i+1} - r_{i+1}$. Then $\phi(\mathrm{Par}(\bigcup \mathcal{C}_{1,i+1}) \cap V(T'_{i+1})) = \{y\}$ and

$$\deg_{G_\nabla}(y, \mathbf{\Psi'}) \geqslant k - |U_i| + \frac{\eta k}{6}\ . \tag{8.21}$$

When all the trees $T_1, \ldots, T_p$ are processed, we define $T' := \{r\} \cup \bigcup \mathcal{C}_M \cup \bigcup_{i=1}^p T'_i$, and set $\mathcal{C}_1 := \bigcup_{i=1}^p \mathcal{C}_{1,i}$. Thus also (a) is satisfied by (8.21) for $i = p$, since $|T'| = |U_p|$. This finishes the proof of the lemma. □

It turns out that our techniques for embedding a tree $T \in \mathbf{trees}(k)$ for Configurations ($\diamond$**2**)–($\diamond$**5**) are very similar. In Lemma 8.18 below we resolve these tasks at once. The proof of Lemma 8.18 follows the same basic strategy for each of the configurations ($\diamond$**2**)–($\diamond$**5**) and deviates only in the elementary procedures of embedding shrubs of T.

Lemma 8.18. *Suppose that we are in Setting 7.4, and one of the following configurations can be found in G:*

a) Configuration ($\diamond$**2**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \rho^3\big)$,

b) Configuration ($\diamond$**3**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \gamma/2, \gamma^3/100\big)$,

c) Configuration ($\diamond$**4**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \gamma/2, \gamma^4/100\big)$, *or*

d) Configuration ($\diamond$**5**) $\Big((\Omega^*)^2, 5(\Omega^*)^9, \varepsilon', 2/(\Omega^*)^3, \frac{1}{(\Omega^*)^5}\Big)$,

Let (T, r) be a rooted tree of order k with a (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. Then $T \subseteq G$.

Proof. First observe that each of the configurations given by a)–d) contains two sets $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}$ and $V_1 \subseteq V(G) \setminus \mathbf{\Psi}$ with

$$\deg^{\min}{}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant 5(\Omega^*)^9 k \ , \tag{8.22}$$

$$\deg^{\min}{}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \varepsilon' k \ . \tag{8.23}$$

For any vertex $z \in W_A \cup W_B$ we define $T(z)$ as the forest consisting of all components of $T - (W_A \cup W_B)$ that contain children of z. Throughout the proof, we write ϕ for the current partial embedding of T into G.

Overview of the embedding procedure. As outlined in Section 8.1.1 the embedding scheme is the same for Configurations ($\diamond$**2**)–($\diamond$**5**). The embedding ϕ is defined in two stages. In Stage 1, we embed $W_A \cup W_B$, all the internal shrubs, all the end shrubs of $\mathcal{S}_A$, and a part[29] of the end shrubs of $\mathcal{S}_B$. In Stage 2 we embed the rest of $\mathcal{S}_B$. Which part of $\mathcal{S}_B$ are embedded in Stage 1 and which part in Stage 2 will be determined during Stage 1. We first give a rough outline of both stages listing some conditions which we require to be met, and then we describe each of the stages in detail.

Stage 1 is defined in $|W_A \cup \{r\}|$ steps. First we map r to any vertex in $\mathbf{\Psi}''$. Then in each step we pick a vertex $x \in W_A$ for which the embedding ϕ has already been defined but such that ϕ is not yet defined for any of the children of x. In this step we embed $T(x)$, together with all the children and grandchildren of x in the knag which contains x. For each $y \in W_B \cap \mathrm{Ch}(x)$, Lemma 8.17 determines a subforest $T'(y) \subseteq T(y)$ which is embedded in Stage 1, and sets $\mathcal{C}_1(y)$ and $\mathcal{C}_2(y)$, which will be embedded in Stage 2.

The embedding in each step of Stage 1 will be defined so that the following properties hold.

(*1) All vertices from W_A are mapped to $\mathbf{\Psi}''$.

(*2) All vertices except for W_A are mapped to $V(G) \setminus \mathbf{\Psi}$.

(*3) For each $y \in W_B$, for each $v \in \mathrm{Par}(V(\bigcup \mathcal{C}_1(y)))$ it holds that

$$\deg_G(\phi(v), \mathbf{\Psi}') \geqslant k + \tfrac{\eta k}{100} - v(T'(y)) \ .$$

(*4) For each $y \in W_B$, for each $v \in \mathrm{Par}(V(\bigcup \mathcal{C}_2(y)))$ it holds that

$$\deg_G(\phi(v), \mathbf{\Psi}') \geqslant \tfrac{k}{2} + \tfrac{\eta k}{100} - v(T'(y) \cup \bigcup \mathcal{C}_1(y)) \ .$$

In Stage 2, we shall utilize properties (*3) and (*4) to embed $T_B^* := \bigcup \mathcal{S}_B - \bigcup_{y \in W_B} T'(y)$. Stage 2 is substantially simpler than Stage 1; this is due to the fact that T_B^* consists only of end shrubs.

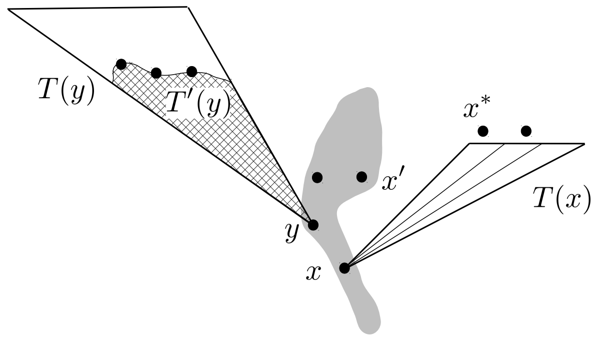

Figure 8.6: Stage 1 of the embedding in the proof of Lemma 8.18. Starting from an already embedded vertex $x \in W_A$ we extend the embedding to (in this order)
(1) all the children $y \in W_B$ of x in the same knag (in grey),
(2) a part $T'(y)$ of the forest $T(y)$,
(3) all the grandchildren $x' \in W_A$ of x in the same knag,
(4) the forest $T(x)$ together with the bordering cut-vertices $x^* \in W_A$.

The embedding step of Stage 1. The embedding step is the same for Configurations (⋄**2**)–(⋄**5**), except for the embedding of internal shrubs. The order of the embedding steps is illustrated in Figure 8.6.

In each step we have picked $x \in W_A$ already embedded in G but such that none of $\mathrm{Ch}(x)$ are embedded. By (*1), or by the choice of $\phi(r)$, we have $\phi(x) \in \mathbf{\Psi}''$. So by (8.22) we have

$$\deg_{G_\nabla}(\phi(x), V_1 \setminus U) \geqslant 5(\Omega^*)^9 k - k. \tag{8.24}$$

First, we embed successively in $|W_B \cap \mathrm{Ch}(x)|$ steps the vertices $y \in W_B \cap \mathrm{Ch}(x)$ together with components $T'(y) \subseteq T(y)$ which will be determined on the way. Suppose that in a certain step we are to embed $y \in W_B \cap \mathrm{Ch}(x)$ and the (to be determined) tree $T'(y)$. Let $F := \bigcup_{i=0}^{2} \mathbf{shadow}^{(i)}_{G_\nabla - \mathbf{\Psi}}(\mathbf{ghost}(U), \frac{\eta k}{10^5})$, where U is the set of vertices used by the embedding ϕ in previous steps, so $|U| \leqslant k$. By Fact 7.1, $|F| \leqslant \frac{10^{10}(\Omega^*)^2}{\eta^2} k$. We embed y anywhere in $(\mathrm{N}_G(\phi(x)) \cap V_1) \setminus F$, cf. (8.22). Note that then (*2) holds for y. We use Lemma 8.17 in order to embed $T'(y) \subseteq T(y)$ (the subtree $T'(y)$ is determined by Lemma 8.17). Lemma 8.17 ensures that (*3) and (*4) hold and that we have $\phi(V(T'(y))) \subseteq V(G) \setminus \mathbf{\Psi}$.

Also, we map the vertices $x' \in W_A \cap \mathrm{Ch}(y)$ to $\mathbf{\Psi}'' \setminus U$. To justify this step, employing (*2), it is enough to prove that

$$\deg(\phi(y), \mathbf{\Psi}'') \geqslant |W_A| \,. \tag{8.25}$$

Indeed, on one hand, we have $|W_A| \leqslant 336/\tau$ by Definition 3.1(c). On the other hand, we have that $\phi(y) \in V_1$, and thus (8.23) applies. We can thus embed x' as planned, ensuring (*1), and finishing the step for y.

Next, we sequentially embed the components $\tilde{T}$ of $T(x)$. In the following, we describe such an embedding procedure only for an internal shrub $\tilde{T}$, with x^* denoting the other neighbour of $\tilde{T}$ in

[29] in the sense that individual shrubs $\mathcal{S}_B$ may be embedded only in part

W_A (cf. (*1)). The case when $\tilde{T}$ is an end shrub is analoguous: actually it is even easier as we do not have to worry about placing x^* well. The actual embedding of $\tilde{T}$ together with x^* depends on the configuration we are in. We shall slightly abuse notation by letting U now denote everything embedded before the tree $\tilde{T}$.

For Configuration ($\diamond$**2**), we use Lemma 8.4 for one tree, namely $\tilde{T} - x^*$, using the following setting: $Q_{\triangleright L8.4} := 1, \gamma_{\triangleright L8.4} := \gamma, \zeta_{\triangleright L8.4} := \rho^3, H_{\triangleright L8.4} := G_{\mathrm{exp}}$, $U_{\triangleright L8.4} := U$, and $U^*_{\triangleright L8.4} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.24)). The child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, the vertices at odd distance from x get embedded in V_1, and the vertices at even distance from x get embedded in V_2. In particular, $\mathrm{Par}_T(x^*)$ gets embedded in V_1. After this, we accomodate x^* in a vertex in $\mathbf{\Psi}'' \setminus U$ which is adjacent to $\phi(\mathrm{Par}_T(x^*))$. This is possible by the same reasoning as in (8.25).

For Configuration ($\diamond$**3**), we use Lemma 8.13 to embed $\tilde{T}$ with the setting $\gamma_{\triangleright L8.13} := \gamma, \delta_{\triangleright L8.13} := \gamma^3/100, U_{\triangleright L8.13} := U$ and $U^*_{\triangleright L8.13} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.24)). Then the child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, vertices of $\tilde{T}$ of odd distance to x (i.e. of even distance to the root of $\tilde{T}$) get embedded in $V_1 \setminus U$, and vertices of even distance get embedded in $V_2 \setminus U$. We extend the embedding by mapping x^* to a suitable vertex in $\mathbf{\Psi}'' \setminus U$ adjacent to $\phi(\mathrm{Par}_T(x^*))$ in the same way as above.

For Configuration ($\diamond$**4**), we use Lemma 8.14 to embed $\tilde{T}$ with the setting $\gamma_{\triangleright L8.14} := \gamma, \delta_{\triangleright L8.14} := \gamma^4/100, U_{\triangleright L8.14} := U$ and $U^*_{\triangleright L8.14} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.24)). The fruit $r'_{\triangleright L8.14}$ in the lemma is chosen as $\mathrm{Par}_T(x^*)$, note that this is indeed a fruit (in $\tilde{T}$) because of Definition 3.1 (i). Then the child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, the vertex $r'_{\triangleright L8.14} = \mathrm{Par}_T(x^*)$ gets embedded in $V_1 \setminus U$, and the rest of $\tilde{T}$ gets embedded in $(\mathfrak{A}' \cup V_2) \setminus U$. This allows us to extend the embedding to x^* as above.

In Configuration ($\diamond$**5**), let $\mathbf{W} \subseteq \mathbf{V}$ denote the set of those clusters, which have at least an $\frac{1}{2(\Omega^*)^5}$-fraction of their vertices contained in the set $U' := U \cup \mathbf{shadow}_{G_{\mathrm{reg}}}(U, k/(\Omega^*)^3)$. We get from Fact 7.1 that $|U'| \leqslant 2(\Omega^*)^4 k$, and consequently $|U' \cup \bigcup \mathbf{W}| \leqslant 4(\Omega^*)^9 k$. By (8.24) we can find a vertex $v \in (\mathrm{N}_G(\phi(x)) \cap V_1) \setminus (U' \cup \bigcup \mathbf{W})$.

We use the fact that $v \notin \mathbf{shadow}_{G_{\mathrm{reg}}}(U, k/(\Omega^*)^3)$ together with inequality (7.39) to see that $\deg_{G_{\mathrm{reg}}}(v, V(G_{\mathrm{reg}}) \setminus U) \geqslant k/(\Omega^*)^3$. Now, since there are only boundedly many clusters seen from v (cf. Fact 4.11), there must be a cluster $D \in \mathbf{V}$ such that

$$\deg_{G_{\mathrm{reg}}}(v, D \setminus U) \geqslant \frac{\gamma^2}{2 \cdot (\Omega^*)^5}|D| \geqslant \gamma^3|D| \ . \tag{8.26}$$

Let C be the cluster containing v. We have $|(C \cap V_1) \setminus U| \geqslant \frac{1}{2(\Omega^*)^5}|C| \geqslant \gamma^3|C|$ because of (7.40) and since $C \notin \mathbf{W}$. Thus, by Fact 2.7, $\big((C \cap V_1) \setminus U, D \setminus U\big)$ is an $2\varepsilon'/\gamma^3$-regular pair of density at least $\gamma^2/2$. We can therefore embed $\tilde{T}$ in this pair using the regularity method. Moreover, by (8.26), we can do so by mapping the child z of x to v. Thus the parent of x^* (lying at even distance to z) will be embedded in $(C \cap V_1) \setminus U$. We can then extend our embedding to x^* as above.

This finishes our embedding of $T(x)$. Note that in all cases we have $\phi(x^*) \in \mathbf{\Psi}''$ and $\phi(V(\tilde{T})) \subseteq V(G) \setminus \mathbf{\Psi}$, as required by (*1) and (*2).

The embedding steps of Stage 2. For $i = 1, 2$, set $Z_i := \bigcup_{y \in W_B} \mathrm{Ch}(T'(y)) \cap \bigcup \mathcal{C}_i(y)$.

First, we embed all the vertices $z \in Z_2$ in $\mathbf{\Psi'}$. By (*2), until now, only vertices of $W_A \cup Z_2$ are mapped to $\mathbf{\Psi'}$, and using (*4) and the properties (c), (k) and (l) of Definiton 3.1, we see that

$$\begin{aligned}\deg_G(\phi(\mathrm{Par}(z)), \mathbf{\Psi'}) &\geqslant \frac{\eta k}{100} + (\frac{k}{2} - \bigcup_{y \in W_B} (T'(y) \cup \bigcup \mathcal{C}_1(y)) \\ &> |W_A| + |Z_2| \ .\end{aligned}$$

So there is space for the vertex z in $\mathbf{\Psi'} \cap \phi(\mathrm{N}_G(\mathrm{Par}(z)))$.

Next, we embed all the vertices $z \in Z_1$ in $\mathbf{\Psi'}$. By (*2), until now, only vertices of $W_A \cup Z_2 \cup Z_1$ are mapped to $\mathbf{\Psi'}$, and by (*3) we have, similarly as above,

$$\deg_G(\phi(\mathrm{Par}(z)), \mathbf{\Psi'}) > |W_A| + |Z_2| + |Z_1| \ .$$

So z can be embedded in $\mathbf{\Psi'} \cap \mathrm{N}_G(\phi(\mathrm{Par}(z)))$ as planned.

Finally, for $z \in Z_1 \cup Z_2$, denote by T_z the component of $\mathcal{C}_1 \cup \mathcal{C}_2$ that contains z. We use Lemma 8.16 to embed the rest of the rooted tree (T_z, z). (Note that our parameters work because of (7.3).) Once all rooted trees (T_z, z), $z \in Z_1 \cup Z_2$ have been processed, we have finished Stage 2 and thus the proof of the lemma. □

8.4.3 Embedding in Configurations ($\diamond$**6**)–($\diamond$**10**)

We follow the schemes outlined in Sections 8.1.2, 8.1.3, 8.1.4, and 8.1.5.

Embedding a tree $T_{\triangleright \mathrm{T}1.3} \in \mathbf{trees}(k)$ using Configurations ($\diamond$**6**), ($\diamond$**7**), ($\diamond$**8**) has two parts: first the internal part of $T_{\triangleright \mathrm{T}1.3}$ is embedded, and then this partial embedding is extended to end shrubs of $T_{\triangleright \mathrm{T}1.3}$ as well. Lemma 8.19 (for configurations ($\diamond$**6**) and ($\diamond$**7**)) and Lemma 8.20 (for configuration ($\diamond$**8**)) are used for the former part, and Lemmas 8.21 and 8.22 (depending on whether we have ($\heartsuit$**1**) or ($\heartsuit$**2**)) for the latter. Lemma 8.23 then puts these two pieces together.

Embedding using Configurations ($\diamond$**9**) and ($\diamond$**10**) is resolved in Lemmas 8.24 and 8.25, respectively.

Lemma 8.19. *Suppose we are in Setting 7.4 and 7.7, and we have one of the following two configurations:*

- *Configuration* ($\diamond$**6**)$(\delta_6, \tilde{\varepsilon}, d', \mu, 1, 0)$, *or*
- *Configuration* ($\diamond$**7**)$(\delta_7, \frac{\eta\gamma}{400}, \tilde{\varepsilon}, d', \mu, 1, 0)$,

with $10^5 \sqrt{\gamma}(\Omega^*)^2 \leqslant \delta_6^4 \leqslant 1$, $10^2 \sqrt{\gamma}(\Omega^*)^3/\Lambda \leqslant \delta_7^3 < \eta^3\gamma^3/10^6$, $d' > 10\tilde{\varepsilon} > 0$, *and* $d'\mu\tau k \geqslant 4 \cdot 10^3$. *Both configurations contain distinguished sets* $V_0, V_1 \subseteq \mathfrak{P}_0$ *and* $V_2, V_3 \subseteq \mathfrak{P}_1$.

Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is a* (τk)*-fine partition of a rooted tree* (T, r) *of order at most* k *such that* $|W_A \cup W_B| \leqslant k^{0.1}$. *Let* T' *be the tree induced by all the cut-vertices* $W_A \cup W_B$ *and all the internal shrubs.*

Then there exists an embedding ϕ *of* T' *such that* $\phi(W_A) \subseteq V_1$, $\phi(W_B) \subseteq V_0$, *and* $\phi(T' - (W_A \cup W_B)) \subseteq \mathfrak{P}_1$.

Proof. For simplicity, let us assume that $r \in W_A$. The case when $r \in W_B$ is similar. The (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ induces a (τk)-fine partition in T'. By Lemma 3.7, the tree T' has an ordered skeleton $(X_0, X_1, \dots, X_m)$ where the X_i are either shrubs or knags (X_0 being a knag).

Our strategy is as follows. We sequentially embed the knags and the internal shrubs in the order given by the ordered skeleton. For embedding the knags we use Lemma 8.4 in Preconfiguration (**exp**), and Lemma 8.7 in Preconfiguration (**reg**). For embedding the internal shrubs, we use Lemmas 8.11 and 8.12 if we have Configurations ($\diamond$**6**), and ($\diamond$**7**), respectively.

Throughout, ϕ denotes the current (partial) embedding of $(X_0, X_1 \dots, X_m)$. In consecutive steps, we extend ϕ. We define auxiliary sets $D_i \subseteq V(G)$ which will serve for reserving space for the roots of the shrubs X_i. So the set $Z_{<i} := \bigcup_{j<i}(\phi(X_j) \cup D_j)$ contains what is already used and what should (mainly) be avoided.

Let $W_{A,i} := W_A \cap V(X_i)$, and $W_{B,i} := W_B \cap V(X_i)$. For each $y \in W_{A,j}$ with $j \leqslant i$ let

$$S_y := (V_2 \cap \mathrm{N}_G(\phi(y))) \setminus Z_{<i},$$

except if the latter set has size $> k$, in that case we choose a subset of size k. This is a target set for the roots of shrubs adjacent to y.

Also, in the case X_i is a shrub, we write r_i for its root, and f_i for the only other vertex neighbouring $W_A \cup W_B$. Note that f_i is a fruit of (X_i, r_i).

The value $h = 6$ or $h = 7$ indicates whether we have configuration ($\diamond$**6**) or ($\diamond$**7**). Define

$$F_i := \mathbf{shadow}_{G-\mathbf{\Psi}}\left(Z_{<i}, \frac{\delta_h k}{4}\right) \cup Z_{<i}\ . \tag{8.27}$$

Define $U_i := F_i$ if we have Preconfiguration (**exp**) (note that in that case we have Configuration ($\diamond$**6**)). To define U_i in case of Preconfiguration (**reg**) we make use of the super-regular pairs $(Q_0^{(j)}, Q_1^{(j)})$ ($j \in \mathcal{Y}$). Set

$$U_i := F_i \cup \bigcup \left\{ Q_1^{(j)}\ :\ j \in \mathcal{Y}, |Q_1^{(j)} \cap F_i| \geqslant \frac{|Q_1^{(j)}|}{2} \right\}\ . \tag{8.28}$$

In either case, we have $|U_i| \leqslant 2|F_i|$.

Finally, set

$$W_i := \mathbf{shadow}_{G-\mathbf{\Psi}}\left(U_i, \frac{\delta_h k}{2}\right) \cup Z_{<i}\ . \tag{8.29}$$

We will now show how to embed successively all X_i. At each step i, our embedding ϕ will have the following properties:

(a) $\phi(W_{A,i}) \subseteq V_1 \setminus F_i$ and $\phi(W_{B,i}) \subseteq V_0$,

(b) for each $y \in W_{A,j}$ with $j \leqslant i$ we have $|S_y \cap \phi(X_i)| \leqslant |S_y \cap D_i| + k^{3/4}$,

(c) $|Z_{<i+1}| \leqslant 2k$,

(d) $D_i \subseteq \mathfrak{P}_1 \setminus (\phi(X_i) \cup Z_{<i})$,

(e) $\phi(X_i - r_i)$ is disjoint from $\bigcup_{j<i} \cup D_j$,

(f) $\phi(f_i) \in V_2 \setminus W_i$ if X_i is a shrub,

(g) $\phi(X_i) \subseteq \mathfrak{P}_1$ if X_i is a shrub.

(We remark that since r_i is not defined for knags X_i, condition (e) means that $\phi(X_i)$ is disjoint from $\bigcup_{j<i} \cup D_j$ for knags X_i.)

It is clear that the first together with the last condition ensures that in step m we have found the desired embedding for T'.

Before we show how to embed each X_i fulfilling the properties above, let us quickly calculate a useful bound. By Fact 7.1 and (c), we have that $|F_i| \leqslant \frac{9\Omega^*}{\delta_h} k$ for all $i \leqslant m$. Thus, using $|U_i| \leqslant 2|F_i|$ and again Fact 7.1 and (c), this shows

$$|W_i| \leqslant \frac{38(\Omega^*)^2}{\delta_h^2} k \,. \tag{8.30}$$

Now suppose we are at step i with $0 \leqslant i \leqslant m$. That is, we have already embedded all X_j with $j < i$, and are about to embed X_i.

First assume that X_i is a knag. Note that if $i \neq 0$, then there is exactly one fruit f_ℓ with $\ell < i$ which neighbours X_i. Set $N_i := \mathrm{N}_G(\phi(f_\ell))$ in this case, and let $N_i := V(G)$ for $i = 0$. We distinguish between the two preconfigurations we might be in.

Suppose first we are in Preconfiguration (**exp**). Recall that then we are in Configuration ($\diamond$**6**).

We use Lemma 8.4 to embed the single tree X_i with the following setting: $\ell_{\triangleright \mathrm{L}8.4} := 1$, $V_{1,\triangleright \mathrm{L}8.4} := V_1$, $V_{2,\triangleright \mathrm{L}8.4} := V_0$, $U^*_{\triangleright \mathrm{L}8.4} := (N_i \cap V_1) \setminus U_i = (N_i \cap V_1) \setminus F_i$, $U_{\triangleright \mathrm{L}8.4} := U_i = F_i$, $Q_{\triangleright \mathrm{L}8.4} := \frac{18\Omega^*}{\delta_6}$, $\zeta_{\triangleright \mathrm{L}8.4} := \delta_6$, and $\gamma_{\triangleright \mathrm{L}8.4} := \gamma$. Note that $U^*_{\triangleright \mathrm{L}8.4}$ is large enough by (f) for ℓ and by (7.49) and (7.53), respectively. Lemma 8.4 gives an embedding of the tree X_i such that $\phi(V_{\mathrm{even}}(X_i)) \subseteq V_1 \setminus F_i$ and $\phi(V_{\mathrm{odd}}(X_i)) \subseteq V_0 \setminus F_i$, which maps the root of X_i to the neighbourhood of its parent's image. Note that this ensures (a) and (e) for step i, and setting $D_i := \emptyset$ we also ensure (c) and (d). Property (b) holds since $V_2 \cap \phi(X_i) = \emptyset$. Since X_i is a knag, (f) and (g) are empty.

Suppose now we are in Preconfiguration (**reg**). Then let $j \in \mathcal{Y}$ be such that $(N_i \cap Q_1^{(j)}) \setminus U_i \neq \emptyset$. Such an index j exists by (f) for ℓ and by (7.49) and (7.53), respectively, if $i \neq 0$, and trivially if $i = 0$. We shall use Lemma 8.7 to embed X_i in $(Q_0^{(j)}, Q_1^{(j)})$. More precisely, we use Lemma 8.7 with $A_{\triangleright \mathrm{L}8.7} := Q_1^{(j)}$, $B_{\triangleright \mathrm{L}8.7} := Q_0^{(j)}$, $\varepsilon_{\triangleright \mathrm{L}8.7} := \tilde{\varepsilon}$, $d_{\triangleright \mathrm{L}8.7} := d'$, $\ell_{\triangleright \mathrm{L}8.7} := \mu k$, $U_A := U_i \cap A$, $U_B := \phi(W_{B,<i}) \cap B$ (then $|U_A| \leqslant |A|/2$ by the definition of U_i and the choice of j).

Lemma 8.7 yields a $(V_{\mathrm{even}}(X_i) \hookrightarrow V_1 \setminus F_i, V_{\mathrm{odd}}(X_i) \hookrightarrow V_0)$-embedding of X_i, which maps the root of X_i to the neighbourhood of its parent's image. Setting $D_i := \emptyset$, we have (a)–(g).

So let us now assume that X_i is a shrub. The parent y of the root r_i of X_i lies in $W_{A,\ell}$ for some $\ell < i$. By (a) for ℓ, we mapped y to a vertex $\phi(y) \in V_1 \setminus F_\ell$. As $\deg_G(\phi(y), V_2) \geqslant \delta_h k$ (by (7.48) and (7.52), respectively), and since $\phi(y) \notin F_\ell$, we have

$$|S_y| \geqslant \frac{3\delta_h k}{4}. \tag{8.31}$$

Using (b) for all j with $\ell \leqslant j < i$, and using that the sets D_j are pairwise disjoint by (d), we see that

$$|S_y \cap \phi(X_0 \cup \ldots \cup X_{i-1})| = |S_y \cap \phi(X_\ell \cup \ldots \cup X_{i-1})| \leqslant |S_y \cap \bigcup_{\ell \leqslant j < i} D_j| + m \cdot k^{3/4} \leqslant |S_y \cap \bigcup_{0 \leqslant j < i} D_j| + m \cdot k^{3/4}.$$

Therefore, and as by (d) and (e), the sets $\phi(X_0 \cup \ldots X_{i-1})$ and $\bigcup_{0 \leqslant j < i} D_j$ are disjoint except for the at most $m \leqslant |W_A \cup W_B| \leqslant k^{0.1}$ roots r_j of shrubs X_j, and since $k \gg 1$, we have

$$|S_y| \geqslant |S_y \cap \phi(X_0 \cup \ldots \cup X_{i-1})| + |S_y \cap \bigcup_{0 \leqslant j < i} D_j| - m \geqslant 2|S_y \cap \phi(X_0 \cup \ldots \cup X_{i-1})| - k^{0.9}.$$

Thus,

$$|S_y \setminus \phi(X_0 \cup \ldots \cup X_{i-1})| \geqslant \frac{|S_y| - k^{0.9}}{2} \overset{(8.31)}{\geqslant} \frac{3\delta_h k}{8} - \frac{k^{0.9}}{2} > \frac{\delta_h k}{3}\ .$$

So for $U^* := S_y \setminus \phi(X_0 \cup \ldots \cup X_{i-1})$ we have that $|U^*| \geqslant \frac{\delta_h k}{3}$. If we have Configuration ($\diamond$**6**) or ($\diamond$**7**) we use Lemma 8.11 or 8.12, respectively, with input $U_{\triangleright L8.11-8.12} := W_i$, $U^*_{\triangleright L8.11-8.12} := U^*$, $L_{\triangleright L8.11-8.12} := |W_{A,i}|$, $\gamma_{\triangleright L8.11-8.12} := \gamma$, the family $\{P_t\}_{\triangleright L8.11-8.12} := \{S_y\}_{y \in W_{A,j}, j<i}$, and the rooted tree (X_i, r_i) with fruit f_i. Further, for Configuration ($\diamond$**6**), set $\delta_{\triangleright L8.11} := \delta_6$, $V_{2,\triangleright L8.11} := V_2$ and $V_{3,\triangleright L8.11} := V_3$ and for Configuration ($\diamond$**7**), set $\delta_{\triangleright L8.12} := \delta_7$, $\ell_{\triangleright L8.12} := 1$, $Y_{1,\triangleright L8.12} := V_2$ and $Y_{2,\triangleright L8.12} := V_3$. The output of Lemma 8.11 or 8.12, respectively, is the extension of our embedding ϕ to X_i, and a set $D_i := C_{\triangleright L8.11-8.12} \subseteq (V_2 \cup V_3) \setminus (W_i \cup \phi(X_i))$ for which properties (a) (which is empty) and properties (b)–(g) hold.

□

Lemma 8.20. *Suppose we are in Setting 7.4 and 7.7 and suppose further we have Configuration* ($\diamond$**8**)$(\delta, \frac{\eta\gamma}{400}, \varepsilon_1, \varepsilon_2, d_1, d_2, \mu_1, \mu_2, h_1, 0)$, *with* $2 \cdot 10^5 (\Omega^*)^6/\Lambda \leqslant \delta^6$, $\delta < \gamma^2\eta^4/(10^{16}(\Omega^*)^2)$, $d_2 > 10\varepsilon_2 > 0$, $d_2\mu_2\tau k \geqslant 4 \cdot 10^3$, *and* $\max\{\varepsilon_1, \tau/\mu_1\} \leqslant \eta^2\gamma^2 d_1/(10^{10}(\Omega^*)^3)$. *Recall that we have distinguished sets* $V_0, \ldots, V_4$ *and a semiregular matching* $\mathcal{N}$.

Let $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *be a* (τk)*-fine partition of a rooted tree* (T, r) *of order at most* k. *Let* T' *be the tree induced by all the cut-vertices* $W_A \cup W_B$ *and all the internal shrubs. Suppose that*

$$v(T') < h_1 - \frac{\eta^2 k}{10^5}\ . \tag{8.32}$$

Then there exists an embedding ϕ *of* T' *such that* $\phi(W_A) \subseteq V_1$, $\phi(W_B) \subseteq V_0$, *and* $\phi(T') \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$.

Proof. We assume that $r \in W_A$. The case when $r \in W_B$ is similar.

Let $\mathcal{K}$ be the set of all knags of the (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T. For each such knag $K \in \mathcal{K}$ set $Y_K := K \cup \mathrm{Ch}_{T'}(K)$. We call the subgraphs Y_K *extended knags*. Set $\mathcal{Y} := \{Y_K : K \in \mathcal{K}\}$ and $W_C := V(\bigcup \mathcal{Y} \setminus \bigcup \mathcal{K})$. Since $W_C \subseteq V(T')$, we clearly have that $|W_C| \leqslant |W_A \cup W_B|$.

Note that the forest $T' - \bigcup \mathcal{Y}$ consists of the set $\mathcal{P}$ of peripheral subshrubs of internal shrubs of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$, and the set $\mathcal{S}$ of principal subshrubs of internal shrubs of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. It is not difficult to observe that there is a sequence $(X_0, X_1, \ldots, X_m)$ such that $X_i = (M_i, Y_i, \mathcal{P}_i)$ such that $M_i \in \mathcal{S}$ and $\mathcal{P}_i \subseteq \mathcal{P}$ for each $i \leqslant m$, and such that the following holds.

(I) $M_0 = \emptyset$ and Y_0 contains r.

(II) $\mathcal{P}_i$ are exactly those peripheral subshrubs whose parents lie in Y_i.

(III) The parent f_i of Y_i lies in M_i (unless $i = 0$).

(IV) The parent r_i of M_i lies in some Y_j with $j < i$ (unless $i = 0$),

(V) $\bigcup_{i \leqslant m} V(M_i \cup Y_i \cup \bigcup \mathcal{P}_i) = V(T')$.

See Figure 8.7 for an illustration.

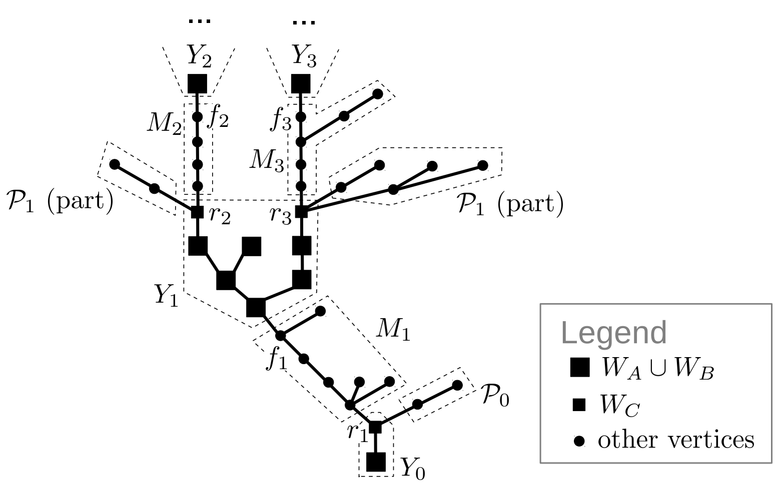

Figure 8.7: An example of a sequence $(X_0, X_1, X_2, X_3, \ldots)$ in Lemma 8.20.

We now successively embed the elements of X_i, except possibly for a part of the subshrubs in $\mathcal{P}_i$. The omitted peripheral subshrubs will be embedded at the very end, after having completed the inductive procedure we are about to describe now.

We shall make use of the following lemmas: Lemma 8.7 (for embedding knags), Lemmas 8.8 and 8.5 (for embedding peripheral subshrubs in $\mathcal{N}$), Lemma 8.12 (for embedding principal subshrubs in $V_3 \cup V_4$).

Throughout, ϕ denotes the current (partial) embedding of T'. In each step i we embed $M_i \cup Y_i$ and a subset of $\mathcal{P}_i$, and denote by $\phi(X_i)$ the image of these sets (as far as it is defined). We also define an auxiliary set $D_i \subseteq V(G)$ which will serve to ensure there is enough space for the roots of the subshrubs M_ℓ with $\ell > i$. Set

$$Z_{<i} := \bigcup_{j<i} (\phi(X_j) \cup D_j).$$

Our plan for embedding the various parts of X_i is depicted in Figure 8.8, which is a refined version of Figure 8.3.

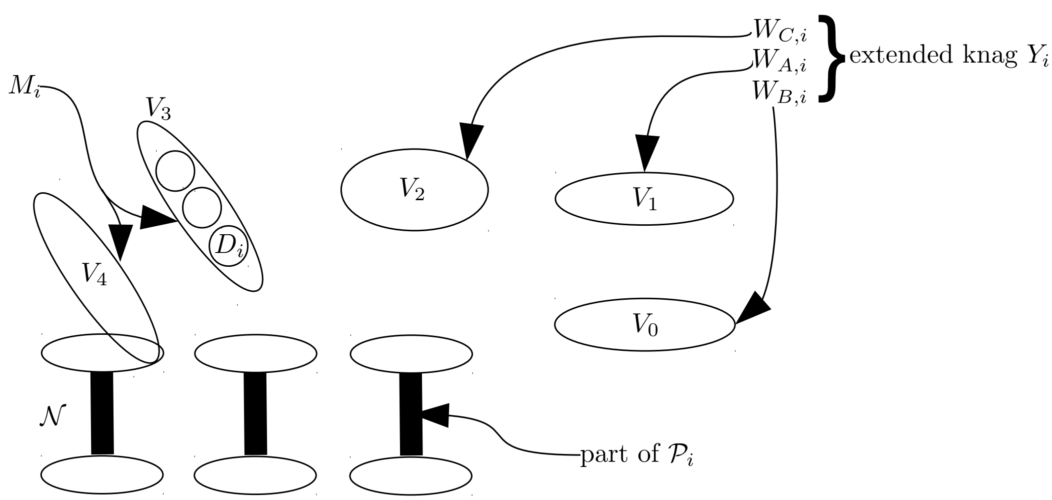

Figure 8.8: Embedding a part of the internal tree in Lemma 8.20.

Let $W_{O,i} := W_O \cap V(Y_i)$ for $O = A, B, C$. For each $y \in W_{C,i}$ let

$$S_y := (V_3 \cap \mathrm{N}_G(\phi(y))) \setminus Z_{<i},$$

except if this set has size more than k, in which case we choose any subset of size k. Similar as in the preceding lemma, this is a target set for the roots of the principal subshrub adjacent to y.

Fix a matching involution $\mathfrak{d}$ for $\mathcal{N}$, and for $\ell = 1, 2$ define

$$F_i^{(\ell)} := Z_{<i} \cup \mathbf{shadow}^{(\ell)}_{G-\mathbf{\Psi}} \left(\mathbf{ghost}_{\mathfrak{d}}(Z_{<i}), \frac{\delta k}{8} \right) . \tag{8.33}$$

We use the super-regular pairs $(Q_0^{(j)}, Q_1^{(j)})$ $(j \in \mathcal{Y})$ to define

$$U_i := F_i^{(2)} \cup \bigcup \left\{ Q_1^{(j)} \ : \ j \in \mathcal{Y}, |Q_1^{(j)} \cap F_i^{(2)}| \geqslant \frac{|Q_1^{(j)}|}{2} \right\} . \tag{8.34}$$

We have

$$|U_i| \leqslant 2|F_i^{(2)}|. \tag{8.35}$$

Finally, for $\ell = 1, 2$ set

$$W_i^{(\ell)} := \mathbf{shadow}^{(\ell)}_{G-\mathbf{\Psi}}\left(U_i, \frac{\delta k}{2}\right) . \tag{8.36}$$

We will now show how to define successively our embedding. At each step i, the embedding ϕ will be defined for $M_i \cup Y_i$ and a subset of $\mathcal{P}_i$, and it will have the following properties:

(a) $\phi(W_{A,i}) \subseteq V_1 \setminus F_i^{(2)}$ and $\phi(W_{B,i}) \subseteq V_0$,

(b) $\phi(W_{C,i}) \subseteq V_2 \setminus F_i^{(1)}$,

(c) $\phi(f_i) \in V_2 \setminus (F_i^{(1)} \cup W_i^{(1)})$,

(d) for each $y \in W_{C,j}$ with $j \leqslant i$ we have $|S_y \cap \phi(X_i)| \leqslant |S_y \cap D_i| + 2k^{3/4}$,

(e) $|Z_{<i+1}| \leqslant 2k$,

(f) $D_i \subseteq V_3 \setminus (\phi(X_i) \cup Z_{<i})$,

(g) $\phi(X_i \setminus (V(M_i) \cap \mathrm{Ch}(W_C)))$ is disjoint from $\bigcup_{j<i} D_j$,[30]

(h) $\phi(X_i) \subseteq \mathfrak{P}_1 \cup \phi(Y_i \cup f_i)$,

(i) if $P \in \mathcal{P}_i$ is not embedded in step i then for its parent $w \in W_C$ we have that $\deg_{G_{\mathcal{D}}}(\phi(w), V_3) \geqslant h_1 - |\phi(X_i) \cap V(\mathcal{N})| - \frac{\eta^2 k}{10^6}$.

Note that for (h), since f_0 is not defined, we assume $\phi(f_0) = \emptyset$.

Before we go on let us remark that (h) together with (f) implies that at each step i we have

$$|Z_{<i} \cap \mathfrak{P}_0| \leqslant 3 \cdot (|W_A| + |W_B|) \overset{\text{D3.1(c)}}{\leqslant} \frac{2016}{\tau} < \frac{\delta k}{8} . \tag{8.37}$$

Also note that by Fact 7.1 and by (e), we have

$$|F_i^{(2)}| \leqslant \frac{65(\Omega^*)^2}{\delta^2} k , \tag{8.38}$$

and

$$|W_i^{(2)}| \leqslant \frac{520(\Omega^*)^4}{\delta^4} k . \tag{8.39}$$

By (b) and by (7.58) we have that $|S_y| \geqslant \frac{7\delta k}{8}$. Now, using (d), (f) and (g), we can calculate similarly as in the previous lemma that at each step i we have

$$|S_y \setminus \bigcup_{\ell \leqslant i} \phi(X_\ell)| \geqslant \frac{3\delta k}{8} . \tag{8.40}$$

[30] Note that $V(M_i) \cap \mathrm{Ch}(W_C)$ contains a single vertex, the root of M_i.

Now assume we are at step i of the inductive procedure, that is, we have already dealt with $X_0, \dots, X_{i-1}$ and wish to embed (parts of) X_i.

We start with embedding M_i, except if $i = 0$, in that case we go directly to embedding Y_0. We shall embed M_i in $V_3 \cup V_4$, except for the fruit f_i, which will be mapped to V_2. The embedding has three stages. First we embed $M_i - M_i(\uparrow f_i)$, then we embed f_i, and finally we embed the forest $M_i(\uparrow f_i) - f_i$. The embedding of $M_i - M_i(\uparrow f_i)$ is an application of Lemma 8.12 analogous to the case of Configuration ($\diamond\mathbf{7}$) in the previous Lemma 8.19. That is, set $Y_{1,\triangleright\mathrm{L}8.12} := V_3$, $Y_{2,\triangleright\mathrm{L}8.12} := V_4$, let

$$U^*_{\triangleright\mathrm{L}8.12} := S_{r_i} \setminus \bigcup_{\ell<i} \phi(X_i),$$

where r_i lies in W_C by (IV), and

$$U_{\triangleright\mathrm{L}8.12} := F_i^{(2)} \cup W_i^{(2)} .$$

Note that

$$|U_{\triangleright\mathrm{L}8.12}| \leqslant \frac{10^3(\Omega^*)^4}{\delta^4} k \leqslant \frac{\delta\Lambda}{2\Omega^*} k,$$

and by (8.40) (which we use for $i-1$), also

$$|U^*_{\triangleright\mathrm{L}8.12}| \geqslant \frac{3\delta k}{8}.$$

The family $\{P_1, \dots, P_L\}_{\triangleright\mathrm{L}8.12}$ is $\{S_y\}_{y\in\bigcup_{j<i} W_{C,j}}$. There is only one tree to be embedded, namely $M_i - M_i(\uparrow f_i)$. It is not difficult to check that all the conditions of Lemma 8.12 are fulfilled. Lemma 8.12 gives an embedding of $M_i - M_i(\uparrow f_i)$ in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$ with the property that $\mathrm{Par}(f_i)$ is mapped to $V_3 \setminus (F_i^{(2)} \cup W_i^{(2)})$. The lemma further gives a set $D' := C_{\triangleright\mathrm{L}8.12}$ of size $v(M_i - M_i(\uparrow f_i))$ such that

$$|S_y \cap \phi(M_i - M_i(\uparrow f_i))| \leqslant |S_y \cap D'| + k^{0.75}$$

for each $y \in \bigcup_{j<i} W_{C,j}$.

Using the degree condition (7.59) we can embed f_i to

$$V_2 \setminus (F_i^{(1)} \cup W_i^{(1)})$$

(recall that (8.37) asserts that only very little space in V_2 is occupied). This ensures (c) for i.

To embed $M_i(\uparrow f_i) - f_i$ we use again Lemma 8.12. The parameters are this time $Y_{1,\triangleright\mathrm{L}8.12} := V_3$, $Y_{2,,\triangleright\mathrm{L}8.12} := V_4$,

$$\begin{aligned} U^*_{\triangleright\mathrm{L}8.12} &:= (\mathrm{N}_G(\phi(f_i)) \cap V_3) \setminus (Z_{<i} \cup \phi(M_i - M_i(\uparrow f_i))) \text{ , and} \\ U_{\triangleright\mathrm{L}8.12} &:= Z_{<i} \cup \phi(M_i - M_i(\uparrow f_i)) \cup D' . \end{aligned}$$

Note that $|U^*_{\triangleright\mathrm{L}8.12}| \geqslant \frac{\delta k}{4}$ by (7.58), by the fact that $\phi(f_i) \notin W_i^{(1)}$, and as $v(T_i) + i < \delta k/8$. The family $\{P_1, \dots, P_L\}_{\triangleright\mathrm{L}8.12}$ is $\{S_y\}_{y\in\bigcup_{j<i} W_{C,j}}$. The trees to be embedded are the components of

$M_i(\uparrow f_i) - f_i$ rooted at the children of f_i. All the conditions of Lemma 8.12 are fulfilled. The lemma provides an embedding in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$. It further gives a set $D'' := C_{\triangleright\mathrm{L}8.12}$ of size $v(M_i(\uparrow f_i)) - 1$ such that

$$|S_y \cap \phi(M_i(\uparrow f_i) - f_i)| \leqslant |S_y \cap D''| + k^{0.75}$$

for each $y \in \bigcup_{j<i} W_{C,j}$. Then $D_i := V_3 \cap (D' \cup D'')$ is such that for each $y \in \bigcup_{j<i} W_{C,j}$,

$$|S_y \cap \phi(M_i)| \leqslant |S_y \cap D_i| + 2k^{0.75} , \tag{8.41}$$

as $S_y \subseteq V_3$ and $\phi(f_i) \notin V_3$. Note that this choice of D_i also ensures (e) for i, and we have by the choices of $U^*_{\triangleright\mathrm{L}8.12}$ and $U_{\triangleright\mathrm{L}8.12}$ in both applications of Lemma 8.12 that

$$D_i \subseteq V_3 \setminus (\phi(M_i) \cup Z_{<i}) \quad \text{and} \quad \phi(X_i \setminus (V(M_i) \cap \mathrm{Ch}(W_C))) \cap \bigcup_{j<i} D_j = \emptyset. \tag{8.42}$$

We now turn to embedding Y_i. Our plan is to use first Lemma 8.7 to embed $Y_i \setminus W_C$ in $(Q_0^{(j)}, Q_1^{(j)})$ for an appropriate index j. After that, we shall show how to embed $W_{C,i}$.

If $i = 0$ then take an arbitrary $j \in \mathcal{Y}$. Otherwise note that by (III), the parent f_i of the root of Y_i lies in M_i. Note that f_i is a fruit in M_i. Let $j \in \mathcal{Y}$ be such that $(\mathrm{N}_G(\phi(f_i)) \cap Q_1^{(j)}) \setminus U_i \neq \emptyset$. Such an index j exists by (7.57) and the fact that $\phi(f_i) \notin W_i^{(1)}$ by (c) for i.

We use Lemma 8.7 with $A_{\triangleright\mathrm{L}8.7} := Q_1^{(j)}$, $B_{\triangleright\mathrm{L}8.7} := Q_0^{(j)}$, $\varepsilon_{\triangleright\mathrm{L}8.7} := \varepsilon_2$, $d_{\triangleright\mathrm{L}8.7} := d_2$, $\ell_{\triangleright\mathrm{L}8.7} := \mu_2 k$, $U_A := U_i \cap A_{\triangleright\mathrm{L}8.7}$, $U_B := Z_{<i} \cap B_{\triangleright\mathrm{L}8.7}$. By the choice of j and the definition of U_i, we find that U_A is small enough, and using (8.37) we see that U_B is also small enough. Lemma 8.7 yields a $(V_{\mathrm{even}}(Y_i - W_C) \hookrightarrow V_1 \setminus F_i^{(2)}, V_{\mathrm{odd}}(Y_i - W_C) \hookrightarrow V_0)$-embedding of $Y_i - W_C$. We clearly see condition (a) satisfied for i.

We now embed successively the vertices of the set $W_{C,i} = \{w_\ell : \ell = 1, \dots, |W_{C,i}|\}$. By the definition of the set W_C, we know that the parent x of w_ℓ lies in $W_{A,i}$. Combining (7.56) with the fact that $\phi(x) \in V_1 \setminus F_i^{(2)}$ by (a) for i, we have that

$$\left|\mathrm{N}_G\left(\phi(x), V_2 \setminus (F_i^{(1)} \setminus Z_{<i})\right)\right| \geqslant \frac{7\delta k}{8} .$$

Thus by (8.37) and since $V_2 \subseteq \mathfrak{P}_0$, we can accommodate w_ℓ in $V_2 \setminus F_i^{(1)}$. This is as desired for (b) in step i.

We now turn to $\mathcal{P}_i$. We will embed a subset of these peripheral subshrubs in $\mathcal{N}$. This procedure is divided into two stages. First we shall aim to embed as many subshrubs as possible in $\mathcal{N}$ in a balanced way, with the help of Lemma 8.8. When it is no longer possible to embed any subshrub in a balanced way in $\mathcal{N}$, we embed in $\mathcal{N}$ as many of the leftover subshrubs as possible, in an unbalanced way. For this part of the embedding we use Lemma 8.5.

By (II) all the parents of the subshrubs in $\mathcal{P}_i$ lie in $W_{C,i}$. For $w_\ell \in W_{C,i}$, let $\mathcal{P}_{i,\ell}$ denote the set of all subshrubs in $\mathcal{P}_i$ adjacent to w_ℓ. In the first stage, we shall embed, successively for $j = 1, \ldots, |W_{C,i}|$, either all or none of $\mathcal{P}_{i,j}$ in a balanced way in $\mathcal{N}$. Assume inductively that

$$\phi\Big(\bigcup_{p<j} \mathcal{P}_{i,p}\Big) \text{ is } (\tau k)\text{-balanced with respect to } \mathcal{N}. \tag{8.43}$$

Construct a semiregular matching $\mathcal{N}_j$ absorbed by $\mathcal{N}$ as follows. Let $\mathcal{N}_j := \{(X_1', X_2') : (X_1, X_2) \in \mathcal{N}\}$, where for $(X_1, X_2) \in \mathcal{N}$ we define (X_1', X_2') as the maximal balanced unoccupied subpair seen from $\phi(w_j)$, i.e., for $b = 1, 2$, we take

$$X_b' \subseteq \big(X_b \cap \mathrm{N}_{G_{\mathrm{reg}}}(\phi(w_j))\big) \setminus \left(\phi(\bigcup_{p<j} \mathcal{P}_{i,p}) \cup \bigcup_{\ell<i} \phi(X_\ell)\right)$$

maximal subject to $|X_1'| = |X_2'|$. If $|V(\mathcal{N}_j)| \geqslant \frac{\eta^2 k}{10^7 \Omega^*}$ then we shall embed $\mathcal{P}_{i,j}$, otherwise we do not embed $\mathcal{P}_{i,j}$ in this step. So assume we decided to embed $\mathcal{P}_{i,j}$. Recall that the total order of the subshrubs in this set is at most τk. Using the same argument as for Claim 8.17.1 we have

$$\left|\bigcup\{X \cup Y \,:\, (X, Y) \in \mathcal{N}, \deg_{G_{\mathcal{D}}}(\phi(w_j), X \cup Y) > 0\right| \leqslant \frac{4(\Omega^*)^2}{\gamma^2} k\,.$$

Thus, there exists a subpair $(X_1', X_2') \in \mathcal{N}_j$ of some $(X_1, X_2) \in \mathcal{N}$ with

$$\frac{|X_1'|}{|X_1|} \geqslant \frac{\frac{\eta^2}{10^7 \Omega^*} k}{\frac{4(\Omega^*)^2}{\gamma^2} k} \geqslant \frac{\gamma^2 \eta^2}{10^8 (\Omega^*)^3}\,. \tag{8.44}$$

In particular, (X_1', X_2') forms a $\frac{2 \cdot 10^8 \varepsilon_1 (\Omega^*)^3}{\gamma^2 \eta^2}$-regular pair of density at least $d_1/2$ by Fact 2.7. We use Lemma 8.8 to embed $\mathcal{P}_{i,j}$ in $\mathcal{M}_{\triangleright \mathrm{L}8.8} := \{(X_1', X_2')\}$. The family $\{f_{CD}\}_{\triangleright \mathrm{L}8.8}$ comprises of a single number $f_{(X_1', X_2')}$ which is the discrepancy of $\bigcup_{p<j} \phi(\mathcal{P}_{i,p})$ with respect to (X_1, X_2). This guarantees that (8.43) is preserved. This finishes the j-th step. We repeat this step until $j = |W_{C,i}|$, then we go to the next stage.

Denote by $\mathcal{Q}_i$ the set of all $P \in \mathcal{P}_i$ that have not been embedded in the first stage. Note that for each $Q \in \mathcal{Q}_i$, with $Q \in \mathcal{P}_{i,j}$, say, and for each $(X_1, X_2) \in \mathcal{N}$ there is a $b_{(X_1,X_2)} \in \{1, 2\}$ such that for

$$O_j := \bigcup_{(X_1,X_2) \in \mathcal{N}} \Big(X_{b_{(X_1,X_2)}} \cap \mathrm{N}_{G_{\mathrm{reg}}}(\phi(w_j))\Big) \setminus \left(\phi(\bigcup_{p<j} \mathcal{P}_{i,p}) \cup \bigcup_{\ell<i} \phi(X_\ell)\right)$$

we have that

$$|O_j| < \frac{\eta^2 k}{10^7 \Omega^*}. \tag{8.45}$$

The fact that O_j is small implies that there an $\mathcal{N}$-cover such that the G_{reg}-neighborhood of w_j restricted to this cover is essentially exhausted by the image of T'.

In the second stage, we shall embed some of the peripheral subshrubs of $\mathcal{Q}_i$. They will be mapped in an unbalanced way to $\mathcal{N}$. We will do this in steps $j = 1, \ldots, |W_{C,i}|$, and denote by $\mathcal{R}_j$ the set of those $\mathcal{P} \subseteq \mathcal{Q}_i$ embedded until step j. At step j, we decide to embed $\mathcal{P}_{i,j}$ if $\mathcal{P}_{i,j} \subseteq \mathcal{Q}_i$ and

$$\deg_{G_{\mathrm{reg}}}\Big(\phi(w_j), V(\mathcal{N}) \setminus \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i)\Big) - |\bigcup \mathcal{R}_{j-1}| \geqslant \frac{\eta^2 k}{10^6} \ . \tag{8.46}$$

Let

$$\tilde{\mathcal{N}} := \left\{ (X,Y) \in \mathcal{N} \ : \ |(X \cup Y) \cap Z_{<i}| < \frac{\gamma^2 \eta^2}{10^9 (\Omega^*)^2} |X| \right\} \ .$$

As by (b) we know that w_j was embedded in $V_2 \setminus F_i^{(1)}$, we have

$$\deg_{G_{\mathrm{reg}}}(\phi(w_j), V(\mathcal{N} \setminus \tilde{\mathcal{N}})) \leqslant \frac{2 \cdot 10^9 (\Omega^*)^2}{\gamma^2 \eta^2} \cdot \frac{\delta k}{8} \leqslant \frac{\eta^2}{10^7} k \ . \tag{8.47}$$

Using (8.45), (8.46) and (8.47), similar calculations as in (8.44) show the existence of a pair $(X,Y) \in \tilde{\mathcal{N}}$ with

$$\deg_{G_{\mathrm{reg}}}(\phi(w_j), (X \cup Y) \setminus (O_j \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i))) - |(X \cup Y) \cap \phi(\bigcup \mathcal{R}_{j-1})| \geqslant \frac{\gamma^2 \eta^2}{10^8 (\Omega^*)^2} |X \cup Y| \ .$$

Then by the definition of $\tilde{\mathcal{N}}$, and setting $Z^+_{<i} := \mathbf{ghost}_{\mathfrak{d}}(Z_{<i})$ we get that

$$\begin{aligned}&\deg_{G_{\mathrm{reg}}}(\phi(w_j), (X \cup Y) \setminus (Z^+_{<i} \cup O_j \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i))) - |(X \cup Y) \cap \phi(\bigcup \mathcal{R}_{j-1})| \\ &\qquad \geqslant \frac{\gamma^2 \eta^2}{10^9 (\Omega^*)^2} |X \cup Y|.\end{aligned}$$

By the definition of O_j, all of the degree counted here goes to one side of the matching edge (X,Y), say to X. So

$$\begin{aligned}\deg_{G_{\mathrm{reg}}}(\phi(w_j), X \setminus (Z^+_{<i} \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i \cup \bigcup \mathcal{R}_{j-1}))) - |Y \cap \phi(\bigcup \mathcal{R}_{j-1})| &\geqslant \frac{\gamma^2 \eta^2}{10^9 (\Omega^*)^2} |X| \qquad (8.48) \\ &\geqslant 12 \frac{\varepsilon_1}{d_1} |X| + \tau k. \qquad (8.49)\end{aligned}$$

We claim that furthermore,

$$|Y \setminus (Z^+_{<i} \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i \cup \bigcup \mathcal{R}_{j-1}))| \geqslant \frac{\gamma^2 \eta^2}{10^{10} (\Omega^*)^2} |Y| \geqslant 12 \frac{\varepsilon_1}{d_1} |Y| + \tau k \ . \tag{8.50}$$

Indeed, otherwise we get by (8.48) that

$$|X \setminus (Z^+_{<i} \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i))| > |Y \setminus (Z^+_{<i} \cup \phi(\bigcup \mathcal{P}_i \setminus \mathcal{Q}_i))| + \frac{\gamma^2 \eta^2}{10^{10} (\Omega^*)^2} |X|,$$

which is impossible by (8.43) and since $|X| \geqslant \mu_1 k$.

Hence, by (8.49) and (8.50), we can embed $\mathcal{P}_{i,j}$ into the unoccupied part (X,Y) using Lemma 8.5 repeatedly.[31]

[31] Recall that the total order of $\mathcal{P}_{i,j}$ is at most τk.

Note that if some $\mathcal{P}_{i,j}$ has not been embedded in either of the two stages, then the vertex w_j must have a somewhat insufficient degree in $\mathcal{N}$. More precisely, employing (8.46) we see that $\deg_{G_{\mathrm{reg}}}(\phi(w_j), V(\mathcal{N})) - |\phi(X_i) \cap V(\mathcal{N})| < \frac{\eta^2 k}{10^6}$. Combined with (7.62), we find that

$$\deg_{G_{\mathcal{D}}}(\phi(w_j), V_3) \geqslant h_1 - |\phi(X_i) \cap V(\mathcal{N})| - \frac{\eta^2 k}{10^6} ,$$

in other words, (i) holds for i.

This finishes step i of the embedding procedure. Recall that the sets V_3 and $V(\mathcal{N})$ are disjoint. Hence, by (a) and (b), the principal subshrubs are the only parts of T' that were embedded in V_3 (and possibly elsewhere). Thus, using (8.42), we see that (f), (g) and (h) are satisfied for i. Also, by (8.41), (d) holds for i.

After having completed the inductive procedure, we still have to embed some peripheral subshrubs. Let us take sequentially those $P \in \mathcal{P}$ which were not embedded. Say w is the parent of P. By (i) we have

$$\deg_{G_{\mathcal{D}}}(\phi(w), V_3 \setminus \mathrm{Im}(\phi)) \geqslant h_1 - |\mathrm{Im}(\phi) \cap V(\mathcal{N})| - |\mathrm{Im}(\phi) \cap V_3| - \frac{\eta^2 k}{10^6} \overset{(8.32)}{\geqslant} \frac{\eta^2 k}{10^6} .$$

An application of Lemma 8.12 in which $Y_{1,\triangleright\mathrm{L}8.12} := V_3$, $Y_{2,\triangleright\mathrm{L}8.12} := V_4$, $U_{\triangleright\mathrm{L}8.12} := \mathrm{Im}(\phi)$, $U^*_{\triangleright\mathrm{L}8.12} := \mathrm{N}_{G_{\mathcal{D}}}(\phi(w)) \cap V_3 \setminus \mathrm{Im}(\phi)$, and $\{P_1, \ldots, P_L\}_{\triangleright\mathrm{L}8.12} := \emptyset$ gives an embedding of P in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$.

By conditions (a), (b), (c) and (h) we have thus found the desired embedding for T'. □

Lemma 8.21. *Suppose we are in Setting 7.4 and 7.7, and that the sets V_0 and V_1 witness Preconfiguration $(\heartsuit\mathbf{1})(2\eta^3 k/10^3, h)$. Suppose that $U \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$. Suppose that $\{x_j\}_{j=1}^{\ell} \subseteq V_0$ and $\{y_j\}_{j=1}^{\ell'} \subseteq V_1$ are sets of mutually distinct vertices. Let $\{(T_j, r_j)\}_{j=1}^{\ell}$ and $\{(T'_j, r'_j)\}_{j=1}^{\ell'}$ be families of rooted trees such that each component of $T_j - r_j$ and of $T'_j - r'_j$ has order at most τk.*

If

$$\sum_j v(T_j) \leqslant \frac{h}{2} - \frac{\eta^2 k}{1000} , \tag{8.51}$$

$$\sum_j v(T_j) + \sum_j v(T'_j) \leqslant h - \frac{\eta^2 k}{1000} , \text{ and} \tag{8.52}$$

$$|U| + \sum_j v(T_j) + \sum_j v(T'_j) \leqslant k , \tag{8.53}$$

then there exist $(r_j \hookrightarrow x_j, V(T_j) \setminus \{r_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T_j and $(r'_j \hookrightarrow y_j, V(T'_j) \setminus \{r'_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T'_j in G, all mutually disjoint.

Proof. The embedding has three stages. In Stage I we embed some components of $T_j - r_j$ (for all $j = 1, \ldots, \ell$) in the parts of $(\mathcal{M}_A \cup \mathcal{M}_B)$-edges which are "seen in a balanced way from x_j".

In Stage II we embed the remaining components of $T_j - r_j$. Last, in Stage III we embed all the components $T'_j - r'_j$ (for all $j = 1, \dots, \ell'$).

Let us first give a bound on the total size of $(\mathcal{M}_A \cup \mathcal{M}_B)$-vertices $C \in \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B)$, $C \subseteq \bigcup \mathbf{V}$ seen from a given vertex via edges of $G_{\mathcal{D}}$. This bound will be used repeatedly.

Claim 8.21.1. Let $v \in V(G)$. Then for $\mathcal{U} := \{C \in \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B) : C \subseteq \bigcup \mathbf{V}, \deg_{G_{\mathcal{D}}}(x, C) > 0\}$ we have

$$|\bigcup \mathcal{U}| \leqslant \frac{2(\Omega^*)^2 k}{\gamma^2} \text{ , and} \tag{8.54}$$

$$|\mathcal{U}| \leqslant \frac{2(\Omega^*)^2 k}{\gamma^2 \pi \mathfrak{c}} \ . \tag{8.55}$$

Proof of Claim 8.21.1. Let $\mathbf{U} \subseteq \mathbf{V}$ be the set of those clusters which intersect $\mathrm{N}_{G_{\mathcal{D}}}(x_j)$. Using the same argument as in the proof of Claim 8.17.1 we get that $|\bigcup \mathbf{U}| \leqslant \frac{2(\Omega^*)^2 k}{\gamma^2}$, i.e. (8.54) holds. Also, (8.55) follows since $\mathcal{M}_A \cup \mathcal{M}_B$ is $(\varepsilon, d, \pi\mathfrak{c})$-semiregular. □

Stage I: We proceed inductively for $j = 1, \dots, \ell$. Suppose that we embedded some components $\mathcal{F}_1, \dots, \mathcal{F}_{j-1}$ of the forests $T_1 - r_1, \dots, T_{j-1} - r_{j-1}$. We write F_{j-1} for the partial images of this embedding. We inductively assume that

$$F_{j-1} \text{ is } \tau k\text{-balanced w.r.t. } \mathcal{M}_A \cup \mathcal{M}_B. \tag{8.56}$$

For each $(A, B) \in \mathcal{M}_A \cup \mathcal{M}_B$ with $\deg_{G_{\mathcal{D}}}(x_j, (A \cup B) \setminus \mathfrak{A}) > 0$ take a subpair (A', B'),

$$A' \subseteq (A \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1} \quad \text{and} \quad B' \subseteq (B \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1} \ ,$$

such that

$$|A'| = |B'| = \min\left\{|(A \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1}|, |(B \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1}|\right\} \ .$$

These pairs comprise a semiregular matching $\mathcal{N}_j$. (Pairs $(A, B) \in \mathcal{M}_A \cup \mathcal{M}_B$ with $\deg_{G_{\mathcal{D}}}(x_j, (A \cup B) \setminus \mathfrak{A}) = 0$ are not considered for the construction of $\mathcal{N}_j$.)

Let $\mathcal{M}_j := \{(A', B') \in \mathcal{N}_j : |A'| > \alpha |A|\}$, for

$$\alpha := \frac{\eta^3 \gamma^2}{10^{10} (\Omega^*)^2}.$$

By Fact 2.7 $\mathcal{M}_j$ is a $(2\varepsilon/\alpha, d/2, \alpha\pi\mathfrak{c})$-semiregular matching.

Claim 8.21.2. We have $|V(\mathcal{M}_j)| \geqslant |V(\mathcal{N}_j)| - \frac{\eta^3 k}{10^9}$.

Proof of Claim 8.21.2. By (8.54), and by Property 4 of Setting 7.4, we have $|V(\mathcal{M}_j)| \geqslant |V(\mathcal{N}_j)| - \alpha \cdot 2 \cdot \frac{2(\Omega^*)^2 k}{\gamma^2}$. □

Let $\mathcal{F}_j$ be a maximal set of components of $T_j - r_j$ such that

$$v(\mathcal{F}_j) \leqslant |V(\mathcal{M}_j)| - \frac{\eta^3 k}{10^9} \;. \tag{8.57}$$

Observe that if $\mathcal{F}_j$ does not contain all the components of $T_j - r_j$ then

$$v(\mathcal{F}_j) \geqslant |V(\mathcal{M}_j)| - \frac{\eta^3 k}{10^9} - \tau k \geqslant |V(\mathcal{M}_j)| - \frac{2\eta^3 k}{10^9} \;. \tag{8.58}$$

Lemma 8.8 yields an embedding of $\mathcal{F}_j$ in $\mathcal{M}_j$. Further the lemma together with the induction hypothesis (8.56) guarantees that the embedding can be chosen so that the new image set F_j is τk-balanced w.r.t. $\mathcal{M}_A \cup \mathcal{M}_B$. We fix this embedding, thus ensuring (8.56) for step i. If $\mathcal{F}_j$ does not contain all the components of $T_j - r_j$ then (8.58) gives

$$|V(\mathcal{M}_j) \setminus F_j| \leqslant \frac{2\eta^3 k}{10^9} \;. \tag{8.59}$$

After Stage I: Let $\mathcal{N}^*$ be a maximal semiregular matching contained in $(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}$ which avoids F_ℓ. We need two auxiliary claims.

Claim 8.21.3. We have

$$\deg^{\max}{}_{G_{\mathcal{D}}}\Big(V_0 \cup V_1, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell \cup \mathfrak{A})\Big) < \frac{\eta^3 k}{10^9} \;.$$

Proof of Claim 8.21.3. Let us consider an arbitrary vertex $x \in V_0 \cup V_1$. By (8.55) the number of $(\mathcal{M}_A \cup \mathcal{M}_B)$-vertices $C \subseteq \bigcup \mathbf{V}$ such that $\deg_{G_{\mathcal{D}}}(x, C) > 0$ is at most $\frac{2(\Omega^*)^2 k}{\gamma^2 \pi \mathfrak{c}}$.

Due to (8.56), we have for each $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) that

$$\Big|(A \cup B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell)\Big| \leqslant \tau k \;. \tag{8.60}$$

Thus summing (8.60) over all $(\mathcal{M}_A \cup \mathcal{M}_B)$-edges (A, B) with $\deg_{G_{\mathcal{D}}}(x, (A \cup B) \setminus \mathfrak{A}) > 0$ we get

$$\deg_{G_{\mathcal{D}}}\Big(x, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell \cup \mathfrak{A})\Big) \leqslant \frac{4(\Omega^*)^2 k}{\gamma^2 \pi \mathfrak{c}} \cdot \tau k \;.$$

The claim now follows by (7.3). □

Claim 8.21.4. Let $j \in [\ell]$ be such that $\mathcal{F}_j$ does not consist of all the components of $T_j - r_j$. Then there exists an $\mathcal{N}^*$-cover $\mathcal{X}_j$ such that $\deg_{G_{\mathcal{D}}}(x_j, \bigcup \mathcal{X}_j) \leqslant \frac{3\eta^3 k}{10^9}$.

Proof of Claim 8.21.4. First, we define an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{R}_j$ as follows. For an $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) let $\mathcal{R}_j$ contain A if

$$|(A \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1}| \leqslant |(B \cap \mathrm{N}_{G_{\mathcal{D}} \cup G_{\nabla}}(x_j))^{\restriction 2} \setminus F_{j-1}| \;,$$

and B otherwise. Observe that by the definition of $\mathcal{N}_j$, we have

$$\deg_{G_{\mathcal{D}}}\Big(x_j, \bigcup \mathcal{R}_j \setminus V(\mathcal{N}_j)\Big) = 0 \;. \tag{8.61}$$

Also, we have $V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j) \subseteq V(\mathcal{N}^*) \cap V(\mathcal{M}_j) \subseteq V(\mathcal{M}_j) \setminus F_j$. In particular, (8.59) gives that

$$\left| V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j) \right| \leqslant \frac{2\eta^3 k}{10^9} . \tag{8.62}$$

Let $\mathcal{X}_j$ be the restriction of $\mathcal{R}_j$ to $\mathcal{N}^*$. We then have

$$\begin{aligned} \deg_{G_{\mathcal{D}}} \left(x_j, \bigcup \mathcal{X}_j \right) &= \deg_{G_{\mathcal{D}}} \left(x_j, V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \right) \\ \text{(by (8.61))} \quad &\leqslant \deg_{G_{\mathcal{D}}} \left(x_j, V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j) \right) + \deg_{G_{\mathcal{D}}} \left(x_j, V(\mathcal{N}_j) \setminus V(\mathcal{M}_j) \right) \\ \text{(by (8.62), Claim 8.21.2)} \quad &\leqslant \frac{3\eta^3 k}{10^9} . \end{aligned}$$

□

For every $j \in [\ell]$ we define $\mathcal{N}_j^* \subseteq \mathcal{N}^*$ as those $\mathcal{N}^*$-edges (A, B) for which we have

$$\left((A \cup B) \setminus \bigcup \mathcal{X}_j \right) \cap \mathfrak{A} = \emptyset .$$

Stage II: We shall inductively for $j = 1, \ldots, \ell$ embed those components of $T_j - r_j$ that are not included in $\mathcal{F}_j$; let us denote the set of these components by $\mathcal{K}_j$. There is nothing to do when $\mathcal{K}_j = \emptyset$, so let us assume otherwise.

We write $\mathbf{L} := \{C \in \mathbf{V} \ : \ C \subseteq \mathbb{L}_{\eta,k}(G)\}$. Let $K \in \mathcal{K}_j$ be a component that has not been embedded yet. We write U' for the total image of what has been embedded (in Stage I, and Stage II so far), combined with U. We claim that x_j has a substantial degree into one of four specific vertex sets.

Claim 8.21.5. At least one of the following four cases occurs.

(U1) $\deg_{G_{\mathcal{D}}} \left(x_j, V(\mathcal{N}_j^*) \setminus \bigcup \mathcal{X}_j \right) - |U' \cap V(\mathcal{N}_j^*)| \geqslant \frac{\eta^2 k}{10^4}$,

(U2) $\deg_{G_{\mathcal{D}}} (x_j, \mathfrak{A} \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

(U3) $\deg_{G_{\nabla}} (x_j, V(G_{\mathrm{exp}}) \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

(U4) $\deg_{G_{\mathcal{D}}} (x_j, \bigcup \mathbf{L} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup U')) \geqslant \frac{\eta^2 k}{10^4}$.

Proof. Write $U'' := (U')^{\restriction 2} = U' \setminus U$. By (7.41), we have

$$\begin{aligned}
\frac{h}{2} &\leqslant \deg_{G_\nabla}(x_j, V_{\mathrm{good}}^{\restriction 2}) \\
&\leqslant \deg_{G_\mathcal{D}}\left(x_j, V(\mathcal{N}_j^*)^{\restriction 2} \setminus \bigcup \mathcal{X}_j\right) + \deg_{G_\mathcal{D}}\left(x_j, \mathfrak{A}^{\restriction 2} \setminus (V(\mathcal{N}_j^*) \cup V(G_{\mathrm{exp}}) \cup \bigcup \mathcal{X}_j)\right) \\
&\quad + \deg_{G_\nabla}\left(x_j, V(G_{\mathrm{exp}})^{\restriction 2}\right) + \deg_{G_\mathcal{D}}\left(x_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V(\mathcal{N}_j^*))\right) \\
&\quad + \deg_{G_\mathcal{D}}\left(x_j, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup \mathfrak{A})\right) + \deg_{G_\mathcal{D}}\left(x_j, \bigcup \mathcal{X}_j\right) \\
\text{(by C8.21.3, C8.21.4)} &\leqslant \deg_{G_\mathcal{D}}\left(x_j, V(\mathcal{N}_j^*) \setminus \bigcup \mathcal{X}_j\right) - |U' \cap V(\mathcal{N}_j^*)| \\
&\quad + \deg_{G_\mathcal{D}}\left(x_j, \mathfrak{A}^{\restriction 2} \setminus (V(\mathcal{N}_j^*) \cup \bigcup \mathcal{X}_j \cup U'')\right) + \deg_{G_\nabla}\left(x_j, V(G_{\mathrm{exp}})^{\restriction 2} \setminus U''\right) \\
&\quad + \deg_{G_\mathcal{D}}\left(x_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V(\mathcal{N}_j^*) \cup U'')\right) \\
&\quad + \frac{4\eta^3 k}{10^9} + |U''| \,.
\end{aligned}$$

The claim follows since $|U''| \leqslant \frac{h}{2} - \frac{\eta^2 k}{1000}$ by (8.51). □

We now now briefly describe how to embed K in each of the cases (**U1**)–(**U4**).

- In case (**U1**) recall that each $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge contains at most one $\mathcal{N}_j^*$-edge. Thus by (8.54) we get that there is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) with

$$\deg_{G_\mathcal{D}}\left(x_j, (V(\mathcal{N}_j^*) \cap (A \cup B)) \setminus \bigcup \mathcal{X}_j\right) - |V(\mathcal{N}_j^*) \cap U' \cap (A \cup B)| \geqslant \frac{\eta^2 k}{10^4} \cdot \frac{\gamma^2}{2(\Omega^*)^2 k} \cdot |A|. \tag{8.63}$$

  Let us fix this edge (A, B), and let (A', B') be the corresponding edge in $\mathcal{N}_j^*$. Suppose without loss of generality that $B \in \mathcal{X}_j$. We can now embed K in (A', B') using Lemma 8.5 with the following input: $C_{\triangleright \mathrm{L}8.5} := A', D_{\triangleright \mathrm{L}8.5} := B', X_{\triangleright \mathrm{L}8.5} := A' \setminus U', X^*_{\triangleright \mathrm{L}8.5} := \mathrm{N}_{G_\mathcal{D}}(x_j, A' \setminus U'), Y_{\triangleright \mathrm{L}8.5} := B' \setminus U', \varepsilon_{\triangleright \mathrm{L}8.5} := \frac{8 \cdot 10^4 (\Omega^*)^2 \varepsilon}{\gamma^2 \eta^2}, \beta_{\triangleright \mathrm{L}8.5} := d/6$. With the help of (8.63), we calculate that $\min\{X_{\triangleright \mathrm{L}8.5}, Y_{\triangleright \mathrm{L}8.5}\} \geqslant |X^*_{\triangleright \mathrm{L}8.5}| \geqslant \frac{\gamma^2 \eta^2 |A|}{4 \cdot 10^4 (\Omega^*)^2} \geqslant 4 \frac{\varepsilon_{\triangleright \mathrm{L}8.5}}{\beta_{\triangleright \mathrm{L}8.5}} |A'|$.

- In Case (**U2**) we embed K using Lemma 8.3 with the following input: $\varepsilon_{\triangleright \mathrm{L}8.3} := \varepsilon', U_{\triangleright \mathrm{L}8.3} := U', U^*_{\triangleright \mathrm{L}8.3} := \mathrm{N}_{G_\mathcal{D}}(x_j, \mathfrak{A} \setminus U'), \ell := 1$.

- In Case (**U3**) we embed K using Lemma 8.4 with the following input: $H_{\triangleright \mathrm{L}8.4} := G_{\mathrm{exp}}, V_{1, \triangleright \mathrm{L}8.4} := V_{2, \triangleright \mathrm{L}8.4} := V(G_{\mathrm{exp}}), U_{\triangleright \mathrm{L}8.4} := U', U^*_{\triangleright \mathrm{L}8.4} := \mathrm{N}_{G_{\mathrm{exp}}}(x_j, V(G_{\mathrm{exp}}) \setminus U'), Q_{\triangleright \mathrm{L}8.4} := 1, \zeta_{\triangleright \mathrm{L}8.4} := \rho, \ell_{\triangleright \mathrm{L}8.4} := 1$.

- In Case (**U4**) we proceed as follows. As $\deg_{G_\mathcal{D}}(x_j, V_{\not\leadsto \mathbf{\Psi}}) < \frac{\eta^2 k}{10^5}$ (cf. Definition 7.17), we have

$$\deg_{G_\mathcal{D}}\left(x_j, \bigcup \mathbf{L} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V_{\not\leadsto \mathbf{\Psi}} \cup U')\right) \geqslant \frac{2\eta^2 k}{10^5} \,.$$

As for (8.63), we use (8.54) to find a cluster $A \in \mathbf{L}$ with

$$\deg_{G_{\mathcal{D}}}\left(x_j, A \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup U')\right) \geqslant \frac{2\eta^2 k}{10^5} \cdot \frac{\gamma^2}{2(\Omega^*)^2 k} \cdot |A| = \frac{\eta^2\gamma^2}{10^5(\Omega^*)^2} \cdot |A|. \quad (8.64)$$

Recall that by the definition of $L_\#$ and $V_{\not\leadsto\mathbf{\Psi}}$, we have that $\deg^{\min}{}_{G_\nabla}(A \setminus (L_\# \cup V_{\not\leadsto\mathbf{\Psi}}), V(G) \setminus \mathbf{\Psi}) \geqslant (1 + \frac{4\eta}{5})k$. Thus at least one of the following subcases must occur for the set $A^* := (\mathrm{N}_{G_{\mathcal{D}}}(x_j) \cap A) \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup U')$:

(**U4a**) For at least $\frac{1}{2}|A^*|$ vertices $v \in A^*$ we have $\deg_{G_\nabla}(v, \mathfrak{A} \setminus U') \geqslant \frac{2\eta k}{5}$.

(**U4b**) For at least $\frac{1}{2}|A^*|$ vertices $v \in A^*$ we have $\deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus U') \geqslant \frac{2\eta k}{5}$.

In case (**U4a**) we embed K using Lemma 8.3. Details are very similar to (**U2**). As for case (**U2b**), let as take an arbitrary vertex $v \in A^*$ with $\deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus U') \geqslant \frac{2\eta k}{5}$. In particular, using (8.54), we find a cluster $B \in \mathbf{V}$ with

$$\deg_{G_{\mathrm{reg}}}(v, B \setminus U') \geqslant \frac{\gamma^2\eta}{10(\Omega^*)^2}|B| \,. \quad (8.65)$$

Map the root r_K of K to v and embed $K - r_K$ in (A, B) using Lemma 8.5[32] with the following input: $C_{\triangleright\mathrm{L}8.5} := B, D_{\triangleright\mathrm{L}8.5} := A, X_{\triangleright\mathrm{L}8.5} := B \setminus U', Y_{\triangleright\mathrm{L}8.5} := A \setminus U', X^*_{\triangleright\mathrm{L}8.5} := \mathrm{N}_{G_{\mathrm{reg}}}(v, B \setminus U'), \beta_{\triangleright\mathrm{L}8.5} := \gamma^2\eta/(10(\Omega^*)^2), \varepsilon_{\triangleright\mathrm{L}8.5} := \varepsilon'$. By (8.64) and (8.65) we see that $X_{\triangleright\mathrm{L}8.5}, Y_{\triangleright\mathrm{L}8.5}$ and $X^*_{\triangleright\mathrm{L}8.5}$ are large enough.

Stage III: In this stage we embed the trees $\{T'_j\}_{j=1}^{\ell'}$. The embedding techniques are as in Stage II. The cover $\mathcal{F}'$ from Definition 7.17 plays the same role as the covers $\mathcal{X}_j$ in Stage II. Observe that $\mathcal{F}'$ is universal whereas the covers $\mathcal{X}_j$ are specific for each vertex x_j. A second simplification is that in Stage III we use the semiregular matching $(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}$ for embedding (in a counterpart of (**U1**)) instead of $\mathcal{N}^*_j$.

Again we proceed inductively for $j = 1, \ldots, \ell$ with embedding the components of $T'_j - r'_j$, which we denote by $\mathcal{K}'_j$. Let $K \in \mathcal{K}'_j$ be a component that has not been embedded yet. We write U' for the total image of what has been embedded (in Stage I, II, and Stage III so far), combined with U and let $U'' = U' \cap \mathfrak{P}_2$. We claim that y_j has a substantial degree into one of four specific vertex sets.

Claim 8.21.6. At least one of the following four cases occurs.

(**U1′**) $\deg_{G_{\mathcal{D}}}\left(y_j, V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}) \setminus (\mathfrak{A} \cup \bigcup \mathcal{F}')\right)$
$- \left|U'' \cap \left(\bigcup \mathcal{F}' \cup (V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}) \setminus \mathfrak{A}\right)\right| \geqslant \frac{\eta^2 k}{10^4}$,

(**U2′**) $\deg_{G_{\mathcal{D}}}(y_j, \mathfrak{A} \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

(**U3′**) $\deg_{G_\nabla}(y_j, V(G_{\mathrm{exp}}) \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

[32] Lemma 8.5 deals with embedding a single tree in a regular pair, whereas $K - r_K$ has several components. We therefore apply the lemma repeatedly for each component.

(**U4′**) $\deg_{G_{\mathcal{D}}}(y_j, \bigcup \mathbf{L} \setminus (L_\# \cup V(G_{\exp}) \cup U')) \geqslant \frac{\eta^2 k}{10^4}$.

Proof. As $y_j \in V_1$, we have that

$$\begin{aligned}
h &\leqslant \deg_{G_\nabla}(y_j, V_{\mathrm{good}}^{\restriction 2}) \\
&\leqslant \deg_{G_{\mathcal{D}}}\Big(y_j, V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}) \setminus (\mathfrak{A} \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) + \deg_{G_{\mathcal{D}}}\Big(y_j, \mathfrak{A}^{\restriction 2} \setminus (V(G_{\exp}) \cup \bigcup \mathcal{F}'\Big) \\
&\quad + \deg_{G_{\mathcal{D}}}(y_j, \bigcup \mathcal{F}') + \deg_{G_{\mathcal{D}}}\Big(y_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\exp}) \cup V(\mathcal{M}_A \cup \mathcal{M}_B)\Big) \\
&\quad + \deg_{G_\nabla}\Big(y_j, V(G_{\exp})^{\restriction 2}\Big) + \deg_{G_{\mathcal{D}}}\Big(y_j, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2})\Big) \\
\text{(by L 7.8)} \quad &\leqslant \deg_{G_{\mathcal{D}}}\Big(y_j, V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}) \setminus (\mathfrak{A} \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) \\
&\quad - \Big|U'' \cap \Big(\bigcup \mathcal{F}' \cup (V((\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}) \setminus \mathfrak{A})\Big) \setminus V(G_{\exp})\Big| \\
&\quad + \deg_{G_{\mathcal{D}}}\Big(y_j, \mathfrak{A}^{\restriction 2} \setminus (U'' \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) + \deg_{G_\nabla}\Big(y_j, V(G_{\exp})^{\restriction 2} \setminus U''\Big) \\
&\quad + \deg_{G_{\mathcal{D}}}\Big(y_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\exp}) \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup U'')\Big) + \frac{2\eta^3 k}{10^3} + \frac{\eta^2 k}{10^5} + |U''| \,.
\end{aligned}$$

The claim follows since $|U''| \leqslant \sum_j T_j + \sum_j T'_j \leqslant h - \frac{\eta^2 k}{1000}$.

□

Cases (**U1′**)–(**U4′**) are treated analogously as Cases (**U1**)–(**U4**).

□

Lemma 8.22. *Suppose we are in Setting 7.4 and 7.7, and that the sets V_0 and V_1 witness Preconfiguration ($\heartsuit$**2**)(h). Suppose that $U \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$, such that $|U| \leqslant k$. Suppose that $\{x_j\}_{j=1}^{\ell} \subseteq V_0 \cup V_1$ are distinct vertices. Let $\{(T_j, r_j)\}_{j=1}^{\ell}$ be a family of rooted trees such that each component of $T_j - r_j$ has order at most τk.*

If $\sum_j v(T_j) \leqslant h - \eta^2 k/1000$ and $|U| + \sum_j v(T_j) \leqslant k$ then there exist disjoint $(r_j \hookrightarrow x_j, V(T_j) \setminus \{r_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T_j in G.

Proof. The proof is contained in the proof of Lemma 8.21. It suffices to repeat the first two stages of the embedding process in the proof. In that setting, we use $h_{\triangleright \mathrm{L}8.21} = 2h$. Note that the condition $\{x_j\} \subseteq V_0$ in the setting of Lemma 8.21 gives us the same possibilities for embedding as the condition $\{x_j\} \subseteq V_0 \cup V_1$ in the current setting (cf. (7.41) and (7.44)). □

Lemma 8.23. *Suppose we are in Setting 7.4 and 7.7, and at least one of the following configurations occurs:*

- *Configuration ($\diamond$**6**)($\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4)}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, h$),*
- *Configuration ($\diamond$**7**)($\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4)}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, h$), or*
- *Configuration ($\diamond$**8**)($\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5)}, \frac{\eta\gamma}{400}, \frac{4\varepsilon}{\mathfrak{p}_1}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\mathfrak{p}_1\pi\mathfrak{c}}{2k}, \frac{\eta^2\nu}{2\cdot 10^4}, h_1, h$).*

Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is a* τk*-fine partition of a rooted tree* (T, r) *of order* k*. If the total order of the end shrubs is at most* $h - 2\frac{\eta^2 k}{10^3}$ *and the total order of the internal shrubs is at most* $h_1 - 2\frac{\eta^2 k}{10^5}$*, then* $T \subseteq G$*.*

Proof. Let T' be the tree induced by all the cut-vertices $W_A \cup W_B$ and all the internal shrubs. Summing up the order of the internal shrub and the cut-vertices, we get that $v(T') < h_1 - \frac{\eta^2 k}{10^5}$. Fix an embedding of T' as in Lemma 8.19 (in configurations ($\diamond$**6**) and ($\diamond$**7**)), or as in Lemma 8.20 (in configuration ($\diamond$**8**)). This embedding now extends to external shrubs by Lemma 8.21 (in Preconfiguration ($\heartsuit$**1**), which can only occur in Configuration ($\diamond$**6**) and ($\diamond$**7**)), or by Lemma 8.22 (in Preconfiguration ($\heartsuit$**2**)). It is important to remember here that by Definition 3.1(l), the total order of end shrubs in $\mathcal{S}_B$ is at most half the size of the total order of end shrubs. □

The next lemma completely resolves Theorem 1.3 in the presence of Configuration ($\diamond$**9**).

Lemma 8.24. *Suppose we are in Setting 7.4 and 7.7, and assume we have Configuration* ($\diamond$**9**)$(\delta, \frac{2\eta^3}{10^3}, h_1, h_2, \varepsilon_1, d_1, \mu_1, \varepsilon_2, d_2, \mu_2)$ *with* $d_2 > 10\varepsilon_2 > 0$*,* $4 \cdot 10^3 \leqslant d_2\mu_2\tau k$*,* $\max\{d_1, \tau/\mu_1\} \leqslant \gamma^2\eta^2/(4 \cdot 10^7(\Omega^*)^2)$*,* $d_1^2/6 > \varepsilon_1 \geqslant \tau/\mu_1$ *and* $\delta k > 10^3/\tau$*.*
Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is a* τk*-fine partition of a rooted tree* (T, r) *of order* k*. If the total order of the internal shrubs of* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is at most* $h_1 - \frac{\eta^2 k}{10^5}$*, and the total order of the end shrubs is at most* $h_2 - 2\frac{\eta^2 k}{10^3}$ *then* $T \subseteq G$*.*

Proof. Let $V_0, V_1, V_2, \mathcal{N}, \{Q_0^{(j)}, Q_1^{(j)}\}_{j \in \mathcal{Y}}$ and $\mathcal{F}'$ witness ($\diamond$**9**). The embedding process has two stages. In the first stage we embed the knags and the internal shrubs of T. In the second stage we embed the end shrubs. The knags will be embedded in $V_0 \cup V_1$, and the internal shrubs will be embedded in $V(\mathcal{N})$. Lemma 8.21 will be used to embed the end shrubs.

The knags of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ are embedded in such a way that W_A is embedded in V_1 and W_B is embedded in V_0. Since no other part of T is embedded in $V_0 \cup V_1$ in the first stage, each knag can be embedded greedily using the minimum degree condition arising from the super-regularity of the pairs $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j \in \mathcal{Y}}$ using the bound on the total order of knags coming from Definition 3.1(c), and using Lemma 8.7 with the following input: $\varepsilon_{\triangleright\text{L}8.7} := \varepsilon_2$, $d_{\triangleright\text{L}8.7} := d_2$, $\ell_{\triangleright\text{L}8.7} := \mu_2 k$, $U_A \cup U_B$ is the image of $W_A \cup W_B$ embedded so far and $\{A_{\triangleright\text{L}8.7}, B_{\triangleright\text{L}8.7}\} := \{Q_0^{(j)}, Q_1^{(j)}\}$, where $j \in \mathcal{Y}$ is arbitrary for the first knag, and for all other knags P has the property that

$$\mathrm{N}_{G_\mathcal{D}}(\phi(\mathrm{Par}(P))) \cap Q_1^{(j)} \setminus U_A \neq \emptyset.$$

The existence of such an index j follows from the fact that

$$\phi(\mathrm{Par}(P)) \in V_2, \tag{8.66}$$

together with condition (7.64). We shall ensure (8.66) during our embedding of the internal shrubs, see below.

We now describe how to embed an internal shrub $T^* \in \mathcal{S}_A$ whose parent $u \in W_A$ is embedded in a vertex $x \in V_1$. Let $w \in V(T^*)$ be the unique neighbor of a vertex from $W_A \setminus \{u\}$ (cf.

Definition 3.1(h)). Let U be the image of the part of T embedded so far. The next claim will be useful for finding a suitable $\mathcal{N}$-edge for accommodating T^*.

Claim 8.24.1. There exists an $\mathcal{N}$-edge (A,B), or an $\mathcal{N}$-edge (B,A) such that

$$\min\left\{|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap(A\setminus U)|,|B\setminus U|\right\}\geqslant 100d_1|A|+\tau k\ .$$

Proof of Claim 8.24.1. For the purpose of this claim we reorient $\mathcal{N}$ so that $V_2(\mathcal{N})\subseteq\bigcup\mathcal{F}'$.

Suppose the claim fails to be true. Then for each $(A,B)\in\mathcal{N}$ we have $|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap(A\setminus U)|<100d_1|A|+\tau k$ or $|B\setminus U|<100d_1|A|+\tau k$. In either case we get

$$|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap A|-|U\cap(A\cup B)|<100d_1|A|+\tau k\ . \tag{8.67}$$

We write $S:=\bigcup\{V(D):D\in\mathcal{D},x\in V(D)\}$. Combining Fact 4.3 and Fact 4.4 we get that

$$|S|\leqslant\frac{2(\Omega^*)^2k}{\gamma^2}\ . \tag{8.68}$$

Let us look at the number

$$\lambda:=\sum_{(A,B)\in\mathcal{N}}\left(|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap A|-|U\cap(A\cup B)|\right)\ . \tag{8.69}$$

For a lower bound on λ, we write $\lambda=|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2|-|U\cap V(\mathcal{N})|$. (Note that $V_2\subseteq V(\mathcal{N})$ as we are in Configuration $(\diamond\mathbf{9})$.) The first term is at least h_1 by (7.63), while the second term is at most $h_1-\frac{\eta^2k}{10^5}$ by the assumptions of the lemma. Thus $\lambda\geqslant\frac{\eta^2k}{10^5}$.

For an upper bound on λ we only consider those $\mathcal{N}$-edges (A,B) for which $\mathrm{N}_{G_{\mathcal{D}}}(x)\cap A\neq\emptyset$. In that case $A\subseteq S$ (cf. 3 of Setting 7.4). Thus, since $\mathcal{N}$ is $(\varepsilon_1,d_1,\mu_1k)$-semiregular we get that

$$|\{(A,B)\in\mathcal{N}\ :\ \mathrm{N}_{G_{\mathcal{D}}}\cap A\neq\emptyset\}|\leqslant\frac{|S|}{\mu_1k}\ . \tag{8.70}$$

Thus,

$$\begin{aligned}\lambda&\leqslant\sum_{(A,B)\in\mathcal{N},\mathrm{N}_{G_{\mathcal{D}}}(x)\cap A\neq\emptyset}\left(|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap A|-|U\cap(A\cup B)|\right)\\
\text{(by (8.67), (8.70))}\quad&\leqslant 100d_1|S|+\frac{|S|}{\mu_1k}\tau k\\
\text{(by (8.68))}\quad&<\frac{\eta^2k}{10^5}\ ,\end{aligned}$$

a contradiction. This finishes the proof of the claim. □

By symmetry we suppose that Claim 8.24.1 gives an $\mathcal{N}$-edge (A,B) such that $\min\left\{|\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap(A\setminus U)|,|B\setminus U|\right\}\geqslant 100d_1|A|+\tau k$. We apply Lemma 8.5 with input $C_{\triangleright\mathrm{L}8.5}:=A$, $D_{\triangleright\mathrm{L}8.5}:=B$ $X_{\triangleright\mathrm{L}8.5}=X^*_{\triangleright\mathrm{L}8.5}:=\mathrm{N}_{G_{\mathcal{D}}}(x)\cap V_2\cap(A\setminus U)$, $Y_{\triangleright\mathrm{L}8.5}:=B\setminus U$, $\varepsilon_{\triangleright\mathrm{L}8.5}:=\varepsilon_1$, $\beta_{\triangleright\mathrm{L}8.5}:=d_1/3$. Then there exists an embedding of T^* in $V(\mathcal{N})\setminus U$ such that w is embedded in V_2. This ensures (8.66).

We remark that there may be several internal shrubs extending from $u \in W_A$. However Claim 8.24.1 and the subsequent application of Lemma 8.5 allows a sequential embedding of these shrubs. This finishes the first stage of the embedding process.

For the second stage, i.e., the embedding of the end shrubs of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$, we first recall that the total order of end shrubs in $\mathcal{S}_A$ is at most $h_2 - 2\frac{\eta^2 k}{10^3}$, and the total order of end shrubs in $\mathcal{S}_B$ is at most $\frac{1}{2}\big(h_2 - 2\frac{\eta^2 k}{10^3}\big)$ by Definition 3.1(l). The embedding is a straightforward application of Lemma 8.21. $\square$

The next lemma resolves Theorem 1.3 in the presence of Configuration ($\diamond$**10**).

Lemma 8.25. *For every $\eta', d', \Omega > 0$ there exists $\tilde{\varepsilon} > 0$ such that for every $\nu' > 0$ there exists k_0 such that the following holds for each $k > k_0$.*
If G is a graph with Configuration $(\diamond\mathbf{10})(\tilde{\varepsilon}, d', \nu' k, \Omega k, \eta')$ then $\mathbf{trees}(k) \subseteq G$.

Lemma 8.25 was basically resolved in [PS12] and we do not give a proof here. See Section 8.1.5 for discussion.

9 Proof of Theorem 1.3

Let $\alpha > 0$ be given. We set
$$\eta := \min\{\frac{1}{30}, \frac{\alpha}{2}\}.$$
We wish to fix further constants as in (7.3). A trouble is that we do not know the right choice of Ω^* and Ω^{**} yet. Therefore we take $g := \lfloor \frac{100}{\eta^2} \rfloor + 1$ and fix suitable constants

$$\eta \gg \frac{1}{\Omega_1} \gg \frac{1}{\Omega_2} \gg \ldots \gg \frac{1}{\Omega_{g+1}} \gg \rho \gg \gamma \gg d \geqslant \frac{1}{\Lambda} \geqslant \varepsilon \geqslant \pi \geqslant \widehat{\alpha} \geqslant \varepsilon' \geqslant \nu \gg \tau \gg \frac{1}{k_0} > 0\ ,$$

where the relations between the parameters are more exactly as follows:

$$
\begin{aligned}
\frac{1}{\Omega_1} &\leqslant \frac{\eta^9}{10^{25}} \ , \\
\frac{1}{\Omega_{j+1}} &\leqslant \frac{\eta^{27}}{10^{67}\Omega_j^{36}} \quad \text{for each } j = 1, \ldots, g \ , \\
\rho &\leqslant \frac{\eta^9}{10^{25}\Omega_{g+1}^5} \ , \\
\gamma &\leqslant \frac{\eta^{18}\rho^{24}}{10^{90}\Omega_{g+1}^{28}} \ , \\
d &\leqslant \min\left\{ \frac{\gamma^2\eta^2}{10^8\Omega_{g+1}^2}, \beta_{\triangleright\mathrm{L}6.1}(\eta_{\triangleright\mathrm{L}6.1} := \eta, \Omega_{\triangleright\mathrm{L}6.1} := \Omega_{g+1}, \gamma_{\triangleright\mathrm{L}6.1} := \gamma) \right\} \ , \\
\frac{1}{\Lambda} &\leqslant \min\left\{ d, \frac{\eta^{24}\gamma^{24}\rho}{10^{96}\Omega_{g+1}^{36}} \right\} \ , \\
\varepsilon &\leqslant \min\left\{ \frac{1}{\Lambda}, \frac{\gamma^2\eta^3 d\rho}{10^{13}\Omega_{g+1}^4}, \tilde{\varepsilon}_{\triangleright\mathrm{L}8.25}(\eta'_{\triangleright\mathrm{L}8.25} := \eta/40, d'_{\triangleright\mathrm{L}8.25} := \gamma^2 d/2, \Omega_{\triangleright\mathrm{L}8.25} := \frac{(\Omega_{g+1})^2}{\gamma^2}) \right\} \ , \\
\pi &\leqslant \min\{\varepsilon, \pi_{\triangleright\mathrm{L}6.1}(\eta_{\triangleright\mathrm{L}6.1} := \eta, \Omega_{\triangleright\mathrm{L}6.1} := \Omega_{g+1}, \gamma_{\triangleright\mathrm{L}6.1} := \gamma, \varepsilon_{\triangleright\mathrm{L}6.1} := \varepsilon)\} \ , \\
\widehat{\alpha} &\leqslant \min\left\{ \pi, \alpha_{\triangleright\mathrm{L}5.6}\left(\Omega_{\triangleright\mathrm{L}5.6} := \Omega_{g+1}, \rho_{\triangleright\mathrm{L}5.6} := \frac{\gamma^2}{4}, \varepsilon_{\triangleright\mathrm{L}5.6} := \pi, \tau_{\triangleright\mathrm{L}5.6} := 2\rho \right) \right\} \ , \\
\varepsilon' &\leqslant \min\left\{ \frac{\widehat{\alpha}^2\gamma^6\rho^2}{10^3\Omega_{g+1}^4}, \varepsilon'_{\triangleright\mathrm{L}6.1}(\eta_{\triangleright\mathrm{L}6.1} := \eta, \Omega_{\triangleright\mathrm{L}6.1} := \Omega_{g+1}, \gamma_{\triangleright\mathrm{L}6.1} := \gamma, \varepsilon_{\triangleright\mathrm{L}6.1} := \varepsilon) \right\} \ , \\
\nu &\leqslant \min\left\{ \frac{\widehat{\alpha}\rho}{\Omega_{g+1}}, \varepsilon', \nu_{\triangleright\mathrm{L}4.14}(\eta_{\triangleright\mathrm{L}4.14} := \alpha, \Lambda_{\triangleright\mathrm{L}4.14} := \Lambda, \gamma_{\triangleright\mathrm{L}4.14} := \gamma, \varepsilon_{\triangleright\mathrm{L}4.14} := \varepsilon', \rho_{\triangleright\mathrm{L}4.14} := \rho) \right\} \ , \\
\tau &\leqslant 2\varepsilon\pi\nu \ , \\
\frac{1}{k_0} &\leqslant \min\left\{ \frac{\gamma^3\rho\eta^8\tau\nu}{10^3\Omega_{g+1}^3}, \frac{1}{k_0^*} \right\} \ ,
\end{aligned}
$$

with

$$
\begin{aligned}
k_0^* := \max\Big\{ & k_{0,\triangleright\mathrm{L}4.14}(\eta_{\triangleright\mathrm{L}4.14} := \alpha, \Lambda_{\triangleright\mathrm{L}4.14} := \Lambda, \gamma_{\triangleright\mathrm{L}4.14} := \gamma, \varepsilon_{\triangleright\mathrm{L}4.14} := \varepsilon', \rho_{\triangleright\mathrm{L}4.14} := \rho), \\
& k_{0,\triangleright\mathrm{L}5.6}(\Omega_{\triangleright\mathrm{L}5.6} := \Omega_{g+1}, \rho_{\triangleright\mathrm{L}5.6} := \frac{\gamma^2}{4}, \varepsilon_{\triangleright\mathrm{L}5.6} := \pi, \tau_{\triangleright\mathrm{L}5.6} := 2\rho, \alpha_{\triangleright\mathrm{L}5.6} := \widehat{\alpha}, \nu_{\triangleright\mathrm{L}5.6} := \frac{2\rho}{\Omega_{g+1}}), \\
& k_{0,\triangleright\mathrm{L}6.1}(\eta_{\triangleright\mathrm{L}6.1} := \eta, \Omega_{\triangleright\mathrm{L}6.1} := \Omega_{g+1}, \gamma_{\triangleright\mathrm{L}6.1} := \gamma, \varepsilon_{\triangleright\mathrm{L}6.1} := \varepsilon, \nu_{\triangleright\mathrm{L}6.1} := \nu), \\
& k_{0,\triangleright\mathrm{L}7.3}(p_{\triangleright\mathrm{L}7.3} := 10, \alpha_{\triangleright\mathrm{L}7.3} := \eta/100), \\
& k_{0,\triangleright\mathrm{L}8.25}(\eta'_{\triangleright\mathrm{L}8.25} := \eta/40, d'_{\triangleright\mathrm{L}8.25} := \gamma^2 d/2, \tilde{\varepsilon}_{\triangleright\mathrm{L}8.25} := \varepsilon, \Omega_{\triangleright\mathrm{L}8.25} := \frac{(\Omega_{g+1})^2}{\gamma^2}, \nu'_{\triangleright\mathrm{L}8.25} := \pi\sqrt{\varepsilon'}\nu) \Big\}.
\end{aligned}
$$

In particular, this gives us a relation between between α and k_0.

Suppose now that $k > k_0$, and $G \in \mathbf{LKS}(n,k,\alpha)$ is a graph, and $T \in \mathbf{trees}(k)$ is a tree. It is our goal to show that $T \subseteq G$.

We follow the plan outlined in Figure 1.3. First, we process the tree T by considering any (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T rooted at an arbitrary root r. Such a partition exists by Lemma 3.4. Let m_1 and m_2 be the total order of internal shrubs and the end shrubs, respectively. For $i = 1,2$ set

$$\mathfrak{p}_i := \frac{\eta}{100} + \frac{m_i}{(1+\frac{\eta}{30})k},$$

and

$$\mathfrak{p}_0 := \frac{\eta}{100}.$$

In particular we have $\mathfrak{p}_i \in [\frac{\eta}{100}, 1]$ for $i = 1,2,3$.

To find a suitable structure in the graph G we proceed as follows. We apply Lemma 4.14 with input graph $G_{\triangleright\text{L4.14}} := G$ and parameters $\eta_{\triangleright\text{L4.14}} := \alpha$, $\Lambda_{\triangleright\text{L4.14}} := \Lambda$, $\gamma_{\triangleright\text{L4.14}} := \gamma$, $\varepsilon_{\triangleright\text{L4.14}} := \varepsilon'$, $\rho_{\triangleright\text{L4.14}} := \rho$, the sequence $(\Omega_j)_{j=1}^{g+1}$, $k_{\triangleright\text{L4.14}} := k$ and $b_{\triangleright\text{L4.14}} := \frac{\rho k}{100\Omega^*}$. The lemma gives a graph $G'_{\triangleright\text{L4.14}} \in \mathbf{LKSsmall}(n,k,\eta)$, and an index $i \in [g]$. Slightly abusing notation, we call this graph still G. Set $\Omega^* := \Omega_i$ and $\Omega^{**} := \Omega_{i+1}$. Now, item (c) of Lemma 4.14 yields a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\text{reg}}, G_{\text{exp}}, \mathfrak{A})$. Let $\mathfrak{c}$ be the size of any cluster in $\mathbf{V}$.

We now apply Lemma 6.1 with parameters $\eta_{\triangleright\text{L6.1}} := \eta$, $\Omega_{\triangleright\text{L6.1}} := \Omega_{g+1}$, $\gamma_{\triangleright\text{L6.1}} := \gamma$, $\varepsilon_{\triangleright\text{L6.1}} := \varepsilon$, $k_{\triangleright\text{L6.1}} := k$, and $\Omega^*_{\triangleright\text{L6.1}} := \Omega^*$. Given the graph G with its sparse decomposition ∇ the lemma gives three $(\varepsilon, d, \pi\mathfrak{c})$-semiregular matchings $\mathcal{M}_A$, $\mathcal{M}_B$, and $\mathcal{M}_{\text{good}} \subseteq \mathcal{M}_A$ which fulfill the assertion either of case **(K1)**, or of **(K2)**. The matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ also define the sets $\mathbb{XA}$ and $\mathbb{XB}$.

The additional features provided by Lemma 4.14 and Lemma 6.1 guarantee that we are in the situation described in Setting 7.4. We apply Lemma 7.3 as described in Definition 7.6; the numbers $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$ are as defined above. This puts us in the setting described in Setting 7.7. We now use Lemma 7.32 to obtain one of the following configurations.

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$,
- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,

- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\pi, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\pi, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$

Depending on the actual configuration Lemma 8.15, Lemma 8.18, Lemma 8.23, Lemma 8.24, or Lemma 8.25 guarantee that $T \subseteq G$. This finishes the proof of the theorem.

10 Concluding remarks

10.1 Theorem 1.3 algorithmically

We now discuss the algorithmic aspects of our proof. That is, we would like to find an algorithm which finds a copy of a given tree $T \in \mathbf{trees}(k)$ in any given graph $G \in \mathbf{LKS}(n,k,\alpha)$ in time $O(n^C)$. Here the degree C of the polynomial is allowed to depend on α, but not on k. It can be verified that each of the steps of our proof — except the extraction of dense spots (cf. Section 4.7) — can be turned into a polynomial time algorithm. The two randomized steps — random splitting in Section 7.2 and the use of the stochastic process Duplicate in Section 8 — can be also efficiently derandomized using a standard technique for derandomizing the Chernoff bound. Let us sketch how to deal with extracting dense spots.

The idea is as follows. Initially, we pretend that G_{exp} consists of the entire bounded-degree part $G - \mathbf{\Psi}$ (cleaned for minimum degree ρk as in (4.8)). With such a supposed sparse classification ∇_1 we go through Lemma 6.1 and Lemma 7.32 (which builds on Lemmas 7.33, 7.34, and 7.35) to obtain a configuration. We now start embedding T as in Section 8. Note that G_{reg} and $\mathfrak{A}$ are absent, and so, the only embedding techniques are those involving $\mathbf{\Psi}$ and G_{exp}. Now, either we embed T, or we fail. The only possible reason for the failure is that we were unable to perform the one-step look-ahead strategy described in Section 4.5 because G_{exp} was not really nowhere-dense. But then we actually localized a dense spot D_1. We get an updated supposed sparse classification ∇_2 in which D_1 is removed from G_{exp} and put in $\mathcal{D}$ (which of course can give rise to G_{reg} or $\mathfrak{A}$). We keep iterating. Since in each step we extract at least $O(k^2)$ edges we iterate the above at most $e(G)/\Theta(k^2) = O(\frac{n}{k})$ times. We are certain to succeed eventually, since after $\Theta(\frac{n}{k})$ iterations we get an honest sparse classification.

It seems that this iterative method is generally applicable for problems which employ a sparse classification.

10.2 Strengthenings of Theorem 1.3

It would be possible to strengthen Theorem 1.3 with not too much extra effort by removing the approximation concerning the number of large vertices. Actually, having approximation on the degrees, one could even prove the theorem with negative approximation on the number of large vertices, in the following form.

Theorem 10.1. *There exists $c > 0$ such that for every $\alpha > 0$ there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$ we have the following. Each n-vertex graph with at least $(\frac{1}{2} - c\alpha)n$ vertices of degree at least $(1 + \alpha)k$ contains each tree of order k.*

To prove Theorem 10.1 the only thing which has to be done — apart from obvious notational changes to the classes $\mathbf{LKS}(n, k, \eta)$, $\mathbf{LKSmin}(n, k, \eta)$, $\mathbf{LKSsmall}(n, k, \eta)$ — is to strengthen Lemma 6.1. An appropriately changed Lemma 6.1 can still provide one of the structures **(K1)** or **(K2)** under the weakened hypothesis. The subsequent steps of the proof then do not have to be modified at all.

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A doctoral thesis entitled *Structural graph theory* submitted by Jan Hladký in September 2012 under the supervision of Daniel Král at Charles University in Prague is based on this paper. The texts of the two works are almost identical.

Symbol index

General index

References

[AAM$^+$] N. Alon, S. Arora, R. Manokaran, D. Moshovitz, and O. Weinstein. Inapproximability of densest κ-subgraph from average case hardness. Available via http://people.csail.mit.edu/dmoshkov/papers/.

[ADL$^+$94] N. Alon, R. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma. *J. Algorithms*, 16:80–109, 1994.

[AFK02] S. Arora, A. Frieze, and H. Kaplan. A new rounding procedure for the assignment problem with applications to dense graph arrangement problems. *Math. Program.*, 92(1):1–36, 2002.

[AKS81] M. Ajtai, J. Komlós, and E. Szemerédi. The longest path in a random graph. *Combinatorica*, 1(1):1–12, 1981.

[AKS95] M. Ajtai, J. Komlós, and E. Szemerédi. On a conjecture of Loebl. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., pages 1135–1146. Wiley, New York, 1995.

[AKS07] N. Alon, M. Krivelevich, and B. Sudakov. Embedding nearly-spanning bounded degree trees. *Combinatorica*, 27(6):629–644, 2007.

[AKSS] M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi. Erdős-Sós conjecture. In preparation.

[ASS09] N. Alon, A. Shapira, and U. Stav. Can a graph have distinct regular partitions? *SIAM J. Discrete Math.*, 23(1):278–287, 2008/09.

[BCL09] C. Borgs, J. Chayes, and L. Lovász. Moments of two-variable functions and the uniqueness of graph limits. *J. Geom. and Func. Anal*, 19:1597–1619, 2009.

[BCLR89] S. N. Bhatt, F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg. Universal graphs for bounded-degree trees and planar graphs. *SIAM J. Discrete Math.*, 2(2):145–155, 1989.

[BCPS10] J. Balogh, B. Csaba, M. Pei, and W. Samotij. Large bounded degree trees in expanding graphs. *Electron. J. Combin.*, 17(1):Research Paper 6, 9, 2010.

[BD96] S. Brandt and E. Dobson. The Erdős–Sós conjecture for graphs of girth 5. *Discr. Math.*, 150:411–414, 1996.

[BJ] O. Barr and R. Johansson. Another Note on the Loebl–Komlós–Sós Conjecture. Research reports no. 22, (1997), Umeå University, Sweden.

[BLW00] C. Bazgan, H. Li, and M. Woźniak. On the Loebl-Komlós-Sós conjecture. *J. Graph Theory*, 34(4):269–276, 2000.

[Bol84] B. Bollobás. The evolution of sparse graphs. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 35–57. Academic Press, London, 1984.

[BS97] I. Benjamini and O. Schramm. Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *Geom. Funct. Anal.*, 7(3):403–419, 1997.

[CLNGS10] B. Csaba, I. Levitt, J. Nagy-György, and E. Szemeredi. Tight bounds for embedding bounded degree trees. In *Fete of combinatorics and computer science*, volume 20 of *Bolyai Soc. Math. Stud.*, pages 95–137. János Bolyai Math. Soc., Budapest, 2010.

[Coo09] O. Cooley. Proof of the Loebl-Komlós-Sós conjecture for large, dense graphs. *Discrete Math.*, 309(21):6190–6228, 2009.

[Dob02] E. Dobson. Constructing trees in graphs whose complement has no $K_{2,s}$. *Combin. Probab. Comput.*, 11(4):343–347, 2002.

[EFLS95] P. Erdős, Z. Füredi, M. Loebl, and V. T. Sós. Discrepancy of trees. *Studia Sci. Math. Hungar.*, 30(1-2):47–57, 1995.

[FdlV79] W. Fernandez de la Vega. Long paths in random graphs. *Studia Sci. Math. Hungar.*, 14(4):335–340, 1979.

[FP87] J. Friedman and N. Pippenger. Expanding graphs contain all small trees. *Combinatorica*, 7(1):71–76, 1987.

[GG67] L. Gerencsér and A. Gyárfás. On Ramsey-type problems. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 10:167–170, 1967.

[Har72] F. Harary. Recent results on generalized Ramsey theory for graphs. In *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)*, pages 125–138. Lecture Notes in Math., Vol. 303. Springer, Berlin, 1972.

[Hax01] P. E. Haxell. Tree embeddings. *J. Graph Theory*, 36(3):121–130, 2001.

[HK95] P. E. Haxell and Y. Kohayakawa. The size-Ramsey number of trees. *Israel J. Math.*, 89(1-3):261–274, 1995.

[HKS] D. Hefetz, M. Krivelevich, and T. Szabó. Sharp threshold for the appearance of certain spanning trees in random graphs. Available via http://web.mat.bham.ac.uk/D.Hefetz/publications.html.

[Hla] J. Hladký. Szemerédi Regularity Lemma and its applications in combinatorics. MSc. Thesis, Charles University in Prague, 2008, available via http://homepages.warwick.ac.uk/∼maskai/papers.html.

[HLT02] P. E. Haxell, T. Luczak, and P. W. Tingley. Ramsey numbers for trees of small maximum degree. *Combinatorica*, 22(2):287–320, 2002. Special issue: Paul Erdős and his mathematics.

[HP] J. Hladký and D. Piguet. Loebl-Komlós-Sós Conjecture: dense case. Manuscript (arXiv:0805:4834).

[JKS12] D. Johannsen, M. Krivelevich, and W. Samotij. Expanders are universal for the class of all spanning trees. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, pages 1539–1551. SIAM, 2012.

[JKV08] A. Johansson, J. Kahn, and V. Vu. Factors in random graphs. *Random Structures Algorithms*, 33(1):1–28, 2008.

[KO09] D. Kühn and D. Osthus. Embedding large subgraphs into dense graphs. In *Surveys in combinatorics 2009*, volume 365 of *London Math. Soc. Lecture Note Ser.*, pages 137–167. Cambridge Univ. Press, Cambridge, 2009.

[Koh97] Y. Kohayakawa. Szemerédi's regularity lemma for sparse graphs. In *Foundations of computational mathematics (Rio de Janeiro, 1997)*, pages 216–230. Springer, Berlin, 1997.

[Kri10] M. Krivelevich. Embedding spanning trees in random graphs. *SIAM J. Discrete Math.*, 24(4):1495–1500, 2010.

[KS83] J. Komlós and E. Szemerédi. Limit distribution for the existence of Hamiltonian cycles in a random graph. *Discrete Math.*, 43(1):55–63, 1983.

[KS96] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 295–352. János Bolyai Math. Soc., Budapest, 1996.

[KSS01] J. Komlós, G. N. Sárközy, and E. Szemerédi. Spanning trees in dense graphs. *Combin. Probab. Comput.*, 10(5):397–416, 2001.

[LP86] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.

[MS] M. Matsumoto and S. Sakamoto. Constructing trees in graphs whose complement has no one-leaved $K_{2,s}$. Preprint 2009.

[Pós76] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Math.*, 14(4):359–364, 1976.

[PS08] D. Piguet and M. J. Stein. Loebl-Komlós-Sós conjecture for trees of diameter 5. *Electron. J. Combin.*, 15(1):Research Paper 106, 11 pp. (electronic), 2008.

[PS12] D. Piguet and M. J. Stein. An approximate version of the Loebl-Komlós-Sós conjecture. *J. Combin. Theory Ser. B*, 102(1):102–125, 2012.

[Sof00] S. N. Soffer. The Komlós-Sós conjecture for graphs of girth 7. *Discrete Math.*, 214(1-3):279–283, 2000.

[Sun07] L. Sun. On the Loebl-Komlós-Sós conjecture. *Australas. J. Combin.*, 37:271–275, 2007.

[SV10] B. Sudakov and J. Vondrák. A randomized embedding algorithm for trees. *Combinatorica*, 30(4):445–470, 2010.

[SW97] J.-F. Saclé and M. Woźniak. A note on the Erdős–Sós conjecture for graphs without C_4. *J. Combin. Theory (Series B)*, 70(2):229–234, 1997.

[Sze78] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[Woź96] M. Woźniak. On the Erdős–Sós conjecture. *J. Graph Theory*, 21(2):229–234, 1996.

[Zha11] Y. Zhao. Proof of the $(n/2 - n/2 - n/2)$ conjecture for large n. *Electron. J. Combin.*, 18(1):Paper 27, 61, 2011.